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The Exact Solution of Time Fractional Drinfeld–Sokolov–Wilson System by Residual Power Series Method

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Abstract—In this investigation, a residual power series technique for the exact solution of system of time-fractional Drinfeld-Sokolov-Wilson is introduced. The main objective of this study is to solve the system of time-Fractional Drinfeld-Sokolov-Wilson by using residual power series method. The proposed technique obtains Taylor expansion of the solution of a system and reproduces the exact solution when the solution is polynomial. An example is given to apply the exact solution and demonstrate the efficiency, accuracy, and applicability of the presented technique. A residual power series (RPS) method as an alternative technique to obtain exact solution. The results reveal that the technique is very effective, straightforward, and simple.

Keywords—Residual power series; Drinfeld–Sokolov–Wilson; Fractional derivative

الملخص— في هذه الدراسة، تم تقديم طريقة بواقي متسلسلة القوى لإيجاد الحل التام لنظام الزمن الكسري لمعادلات درنفلد- سكولوف- ولسون. الغرض الرئيسي من هذه الورقة هو الحصول على الحل التام لنظام الزمن الكسري درينفيلد - سوكولوف - ويلسون باستخدام طريقة بواقي متسلسلة القوى. التقنية المقترحة تحصلت على توسع لمتسلسلة تايلور لحل النظام وتعيد إيجاد الحل التام عندما يكون الحل متعدد الحدود. تستخدم هذه الطريقة لتوليد حلول غير خطية لنظام الزمن الكسري لمعادلات درنفلد- سكولوف- ولسون على شكل متسلسلة متقاربة بشكل سريع ومن ثم إيجاد الحل التام. تم تقديم مثال لتطبيق الحل الدقيق وإدراجه لإثبات كفاءة التقنية المقدمة ودقتها وقابليتها للتطبيق. طريقة بواقي متسلسلة القوى تعتبر كتقنية بديلة للحصول على الحل التام. أظهرت النتائج أن التقنية فعالة للغاية ومباشرة وبسيطة.

الكلمات المفتاحية— بواقي متسلسلة القوى؛ درنفلد–سكولوف–ولسون؛ التفاضل الكسري

1. Introduction

Many phenomena in engineering and applied sciences can be described successfully by developing the models using fractional calculus, i.e., the theory of derivatives and integrals of non-integer order [1,2]. Thus, appearances of fractional order derivatives make the study more interesting and challenging. Fractional differential equations (FDEs) have gained importance and popularity, mainly due to their demonstrated applications in science and engineering. For example, these equations are increasingly used to model problems in fluid flow, rheology, diffusion, relaxation, electric networks, polymer physics, chemical physics, propagation of seismic waves and many other physical processes.

Considerable attention has been given to the solution of fractional ordinary differential equations (ODEs), fractional integral equations (IEs) and fractional partial differential equations (PDEs) of physical interest. Since most fractional differential equations defined in the conformable fractional derivative do not have exact analytical solutions, therefore approximation and numerical techniques are used extensively. Some analytical methods are used such as Laplace transform method [3]; Fourier transform method [4]; vibrational iteration method and green function method [5]. They relatively provide an analytical approximate solution to linear and nonlinear FDEs.

In this work, we present the residual power series (RPS) method [6] as an alternative technique to obtain analytic solutions of different types of fractional linear and nonlinear partial differential equations applied in mathematics, physics and engineering.

The RPS method is effective and easy to construct power series solution for strongly linear and nonlinear equations without linearization, perturbation, or discretization. Unlike classical power series method, the RPS method does not need to compare the coefficients of the corresponding terms and recursion relation is not required. This method computes the coefficients of the power series by a chain of algebraic equations of one or more variables. In fact, the RPS method is an alternative procedure for obtaining analytic solutions for partial differential equations of fractional order. By error concept, we get a series solution; in practice a truncated series solution. Moreover, the obtained solutions and all their time-fractional derivatives are applicable for each arbitrary point and each multidimensional variable in the given domain. On the other aspect as well, the RPS method does not require any converting while switching from low-order to high-order, as a result the method can be applied directly to a given system by choosing an appropriate initial guesses approximation.

The main objective of this paper is to solve the generalized Drinfeld-Sokolov-Wilson system (DSWS) coupled nonlinear partial differential equations (NPDEs) by RPS method using conformable definition [6]. This system is one of the universal models proposed by Drinfeld and Sokolov (1981) and Wilson (1985). The coupled of (NPDEs) are widely used to describe complex phenomena in various sciences, especially in the physical sciences. Finding explicit and exact solutions, in particular, solitary wave solutions of nonlinear evolution equations in mathematics and physics play important role in nonlinear science.

Now, we consider the following time-fractional Drinfeld-Sokolov-Wilson system

$$\frac{\partial^{\propto} u(x,t)}{\partial t^{\propto}} + av\frac{\partial v(x,t)}{\partial x} = 0,$$

$$\frac{\partial^{\propto} v(x,t)}{\partial t^{\propto}} + b\frac{\partial^{3} v(x,t)}{\partial t^{3}} + \gamma u\frac{\partial v}{\partial x} + \varepsilon v\frac{\partial u(x,t)}{\partial x} = 0.$$

Subject to the initial conditions

$$u(x,0) = x\,, \qquad\qquad v(x,0) = -x\,.$$

where a, b, γ, ε are nonzero parameters and $\propto$ is the fractional derivative with $0 \leq\propto\leq 1$. Note that for $\propto= 1$, the system is reduced to the standard Drinfeld-Sokolov-Wilson system.

In the next section, some definitions and theorems regard conformable derivative and fractional power series are presented. Detailed derivation of the RPS solution of the fractional DSW system has been discussed in section 3. In section 4, we apply our technique of the RPS to find out series solution for DSWS subject to given initial condition. Then the discussions on obtained findings and main conclusion are demonstrated.

2. Preliminaries

In this section, the definition of conformable fractional derivative, definition of Residual power series (RPS) and some of theorems of power series are presented.

Definition 1 [6]: given a function $f: [0, \infty) \to \mathbb{R}$ then the conformable fractional derivative of the f of order α is defined by:

$$f^{\alpha}(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}.$$

for all $t > 0$, $\alpha \in (0, a)$ and $\lim_{t\to 0} f^{\alpha}(t)$ exists then defined $f^{\alpha}(0) = \lim_{t\to 0} f^{\alpha}(t)$.

Definition 2 [2]: a power series expansion of the form

$$\sum_{m=0}^{\infty} C_m (t - t_0)^{m\alpha} = C_0 + C_1 (t - t_0)^{\alpha} + C_2 (t - t_0)^{2\alpha} + \cdots .$$

For all $\mathrm{m} - 1 < \alpha < m$, $t \leq t_0$ is called fractional power series (FPS) about $t = t_0$, where t is a variable and $\mathrm{C_m}$ are constants called the coefficients of the series.

Theorem 1 [7]: suppose that f has a FPS representation at $t_0 = 0$ of the form:

$$f(t) = \sum_{m=0}^{\infty} C_m t^{\alpha m} \quad ,0 < t < \mathcal{R}^{1/\alpha} \;,\mathcal{R} > 0.$$

Moreover, suppose that f is an infinitely conformable α −differentiable function, for some $0 \leq m - 1 < \alpha < m$, in a neighborhood of point $t_0 = 0$, then the coefficients C_m in will take the form $C_m = \frac{f^{(\alpha m)}(0)}{\alpha^m \, m!}$.

Where $f^{(\alpha m)}$ means the application of the conformable fractional derivative m time.

Theorem 2 [6]: for the FPS $\sum_{n=0}^{\infty} c_n t^{n\alpha}$, $t \geq 0$, there are only three possibilities:

1) The series converges only when $t = 0$.
2) The series converges for each $t \geq 0$.
3) There is a positive real number $\mathcal{R}$ such that the series converges whenever $0 \leq t < \mathcal{R}$ and diverges whenever $t \geq \mathcal{R}$, where $\mathcal{R}$ is a non-negative real number represent the radius of convergence.

Definition 3 [6]: a power series of the form $\sum_{m=0}^{\infty} f_m(x) t^{m\alpha}$ is called a multiple fractional power series about $t_0 = 0$ where t is a variable and $f_m(x)$ are functions of x called the coefficient of the series.

Theorem 3 [7]: Suppose that $U(x,t)$ has a multiple FPS representation at $t_0 = 0$ of the form $U(x,t) = \sum_{m=0}^{\infty} I_m(x) t^{\alpha m}$, $0 \le m-1 < \alpha < m$, $0 \le t \le \mathcal{R}^{1/\alpha}$.

If $U_t^{\alpha m}(x,t), m = 0,1,2 \dots.,$ are continues on $I \times \left(0, \mathcal{R}^{\frac{1}{\alpha}}\right)$ then $I_m(x) = \frac{U_t^{\alpha m}(x,0)}{\alpha^m m!}$.

3. Methodology of the RPS for solving time-fractional (DSW) system by conformable definition:

Consider the time-fractional DSW system:

$$\begin{cases} D_t^{\alpha} U + aVV_x = 0 \,, \\ D_t^{\alpha} V + bV_{xx} + \gamma U V_x + \varepsilon V U_x = 0 \,. \end{cases} \quad (1)$$

The initial conditions:

$$\begin{cases} U(x,0) = I_m(x), \\ V(x,0) = J_m(x). \end{cases} \quad (2)$$

The RPS method assume the solution of the Equation (1) as a fractional power series about the initial point $t_0 = 0$ as follows:

$$\begin{cases} U(x,t) = \sum_{m=0}^{\infty} I_m(x) \frac{t^{\alpha m}}{\alpha^m m!} \,, \\ V(x,t) = \sum_{m=0}^{\infty} J_m(x) \frac{t^{\alpha m}}{\alpha^m m!} \,. \end{cases} \quad (3)$$

Where $0 < \alpha \le 1$, $x \epsilon I$; $0 \le t < \mathcal{R}^{\frac{1}{\alpha}}$, obviously $U(x,t)$ and $V(x,t)$ satisfy the Equation (2), hence we can obtain the initial guess approximation of $U(x,t)$ and $V(x,t)$:

$$\begin{cases} U_0(x,0) = I_0(x) = I(x), \\ V_0(x,0) = J_0(x) = J(x). \end{cases} \quad (4)$$

So, equation (3), could be reformulated as:

$$\begin{cases} U(x,t) = I(x) + \sum_{m=1}^{\infty} I_m \frac{t^{\alpha m}}{\alpha^m m!} \,, \\ V(x,t) = J(x) + \sum_{m=1}^{\infty} J_m \frac{t^{\alpha m}}{\alpha^m m!} \,. \end{cases} \quad (5)$$

Then K^{th} truncated series $U_k(x,t)$ and $V_k(x,t)$

$$\begin{cases} U_k = I(x) + \sum_{m=1}^{k} I_m \frac{t^{\alpha m}}{\alpha^m m!}, \\ V_k = J(x) + \sum_{m=1}^{k} J_m \frac{t^{\alpha m}}{\alpha^m m!} \,, \qquad k = 1,2,3,\dots. \end{cases} \quad (6)$$

Now we define the residual function Res_U and Res_V for Equation (1) as follows:

$$\begin{cases} Res_U(x,t) = D_t^{\alpha} U + aVV_x, \\ Res_V(x,t) = D_t^{\alpha} V + bV_{xxx} + \gamma U V_x + \varepsilon V U_x. \end{cases} \quad (7)$$

Therefore *the* K^{th} truncated residual function are:

$$\begin{cases} Res_{U,k}(x,t) = D_t^{\alpha} U_k + aV_k V_k^{(x)}, \\ Res_{V,k}(x,t) = D_t^{\alpha} U_k + bV_k^{(xxx)} + \gamma U_k V_k^{(x)} + \varepsilon V_k U_k^{(x)}. \end{cases} \tag{8}$$

As stated by Hammad and Khalil [4], $Res(x,t) = 0$ and $\lim_{k\to\infty} Res_k(x,t) = Res(x,t)$ for each $x \in I$ and $0 \le \mathrm{t} < \mathcal{R}^{\frac{1}{\alpha}}$.

Therefore , $\frac{\partial^{\alpha r}}{\partial t^{\alpha r}} Res(x,t) = 0$ since the fractional derivative of a constant function in the conformable since is zero, and in the meantime the fractional derivative $\frac{\partial^{\alpha r}}{\partial t^{\alpha r}}$ of $Res(x,t)$ and $Res_k(x,t)$ are matching at $t = t_0$ for each $r = 0,1,2, \dots$.

Now, if we substitute $t_0 = 0, r = k-1$ we obtain

$$\begin{cases} \frac{\partial^{\alpha(k-1)}}{\partial t^{\alpha(k-1)}} Res_{I,K}(x,0) = 0, \\ \frac{\partial^{\alpha(k-1)}}{\partial t^{\alpha(k-1)}} Res_{J,K}(x,0) = 0. \end{cases} \tag{9}$$

4. Exact solution of system of time –fractional Drinfeld–Sokolov–Wilson

In this section, we will find the N^{th} term and the exact solutions for the Drinfeld-Skolov-Wilson (DSW) the test problem consists of the equation (1) with $a = 3,\ b = 2,\ \gamma = 2$ and $\varepsilon = 1$ we get:

$$\begin{cases} \frac{\partial^{\alpha} u}{\partial t^{\alpha}} + 3v \frac{\partial v}{\partial x} = 0, \\ \frac{\partial^{\alpha} v}{\partial t^{\alpha}} + 2\frac{\partial^3 v}{\partial x^3} + 2u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial x} = 0. \end{cases} \tag{10}$$

The initial conditions:

$$\begin{cases} u(x,0) = x, \\ v(x,0) = -x. \end{cases} \tag{11}$$

Where $0 < \alpha \le 1, x \in I, 0 \le t < \mathcal{R}^{1/\alpha}$.

The RPS method assumes the solution of the system:

$$\begin{cases} u(x,t) = x \sum_{n=0}^{\infty} I_n(x) \frac{t^{n\alpha}}{\alpha^n n!}, \\ v(x,t) = \sum_{n=0}^{\infty} J_n(x) \frac{t^{n\alpha}}{\alpha^n n!}. \end{cases} \tag{12}$$

$$\begin{cases} u_0(x,0) = I(x) = x, \\ v_0(x,0) = J(x) = -x. \end{cases} \tag{13}$$

It is easy to get the 1^{st} truncated series solution by setting $n = 1$

$$\begin{cases} u_1(x,t) = x - x\left(\frac{3t^\alpha}{\alpha}\right), \\ v_1(x,t) = -x + x\left(\frac{3t^\alpha}{\alpha}\right). \end{cases} \tag{14}$$

The 2^{nd} truncated series solutions are represented in the following forms:

$$\begin{cases} u_2(x,t) = x - x\left(\frac{3t^\alpha}{\alpha}\right) + x\left(\frac{3t^\alpha}{\alpha}\right)^2, \\ v_2(x,t) = -x + x\left(\frac{3t^\alpha}{\alpha}\right) - x\left(\frac{3t^\alpha}{\alpha}\right)^2. \end{cases} \tag{15}$$

The 3^{rd} truncated series solutions take the forms:

$$\begin{cases} u_3(x,t) = x - x\left(\frac{3t^\alpha}{\alpha}\right) + x\left(\frac{3t^\alpha}{\alpha}\right)^2 - x\left(\frac{3t^\alpha}{\alpha}\right)^3, \\ v_3(x,t) = -x + x\left(\frac{3t^\alpha}{\alpha}\right) - x\left(\frac{3t^\alpha}{\alpha}\right)^2 + x\left(\frac{3t^\alpha}{\alpha}\right)^3. \end{cases} \tag{9}$$

The 4^{th} truncated series solutions take the forms:

$$\begin{cases} u_4(x,t) = x - x\left(\frac{3t^\alpha}{\alpha}\right) + x\left(\frac{3t^\alpha}{\alpha}\right)^2 - x\left(\frac{3t^\alpha}{\alpha}\right)^3 + x\left(\frac{3t^\alpha}{\alpha}\right)^4, \\ \\ v_4(x,t) = -x + x\left(\frac{3t^\alpha}{\alpha}\right) - x\left(\frac{3t^\alpha}{\alpha}\right)^2 + x\left(\frac{3t^\alpha}{\alpha}\right)^3 - x\left(\frac{3t^\alpha}{\alpha}\right)^4. \end{cases} \tag{10}$$

Then the $\mathrm{N^{th}}$ solution:

$$\begin{cases} u_n(x,t) = x(-1)^n\left(\frac{3t^\alpha}{\alpha}\right)^n, \\ v_n(x,t) = x(-1)^{n+1}\left(\frac{3t^\alpha}{\alpha}\right)^n. \end{cases} \qquad n = 0,1,2,\ldots \tag{11}$$

Then the Exact solution is in Taylor series form:

$$\begin{cases} u_n(x,t) = x\ \sum_{n=0}^{\infty}(-1)^n\left(\frac{3t^\alpha}{\alpha}\right)^n, \\ v_n(x,t) = x\ \sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{3t^\alpha}{\alpha}\right)^n. \end{cases} \tag{12}$$

5. Results and Discussions

The formulation presented in Section 2 provides highly accurate solutions for the problems demonstrated in Section 3. We have used MatLab for most of our computational work. In Tables 1 and 2, we have presented absolute errors for $u(x, t)$ and $v(x, t)$ at a spatial domain $0.1 \leq x \leq 1$ for $t = 0.01$. The convergence of the RPS solution is given, through third and fourth order absolute errors. Here we observe that the RPS solution converges rapidly with increasing order of approximation. Furthermore, from figures 1and 2 it is evident that the RPS results are nearly identical to the numerical results. Here the results are very consistent with the increasing time. Three-dimensional surface graphs are used to illustrate the dynamical behavior of the earned results. Through graphical illustrations, it can be noticed that various forms of traveling wave structures are obtained for the time fractional nonlinear DSW system using the RPS method.

Table 1. Comparison of fourth order and third order errors of $u\ (x,\ t)$ corresponding to the numerical solution at time t = 0.01, and 0.1≤ x ≤1.

x	Fourth order absolute error	Third order absolute error
0.1	0.30×10^{-2}	0.30×10^{-2}
0.2	0.60×10^{-2}	0.60×10^{-2}
0.3	0.89×10^{-2}	0.90×10^{-2}
0.4	$1.19\ \times 10^{-2}$	1.20×10^{-2}
0.5	1.48×10^{-2}	1.50×10^{-2}
0.6	1.77×10^{-2}	1.80×10^{-2}
0.7	2.06×10^{-2}	2.10×10^{-2}
0.8	2.34×10^{-2}	2.40×10^{-2}
0.9	2.63×10^{-2}	2.70×10^{-2}
1.0	2.91×10^{-2}	0.30×10^{-1}

Table 2. Comparison of fourth order and third order errors of $v(x,\ t)$ corresponding to the numerical solution at time t = 0.01, and 0.1≤ x ≤1

x	Fourth order absolute error	Third order absolute error
0.1	0.30×10^{-2}	0.30×10^{-2}
0.2	0.60×10^{-2}	0.60×10^{-2}
0.3	0.89×10^{-2}	0.90×10^{-2}
0.4	$1.19\ \times 10^{-2}$	1.20×10^{-2}
0.5	1.48×10^{-2}	1.50×10^{-2}
0.6	1.77×10^{-2}	1.80×10^{-2}
0.7	2.06×10^{-2}	2.10×10^{-2}
0.8	2.34×10^{-2}	2.40×10^{-2}
0.9	2.63×10^{-2}	2.70×10^{-2}
1.0	2.91×10^{-2}	0.30×10^{-1}

6. Conclusion

In this paper we presented the analytical iterative technique based on the residual power series (RPS) is proposed to obtain an exact solution to a nonlinear time-fractional Drinfeld-Sokolov-Wilson (DSW) system. By using this method, the solutions in the form of generalized Taylor series are obtained. It has been found that the construction of this recent RPS method possesses in general a very rapid convergent series due to the embedded generalized Taylor series. The RPS method is promising technique based on its simplicity and accuracy and it is considered an additive tool for the field of fractional theory and computations.

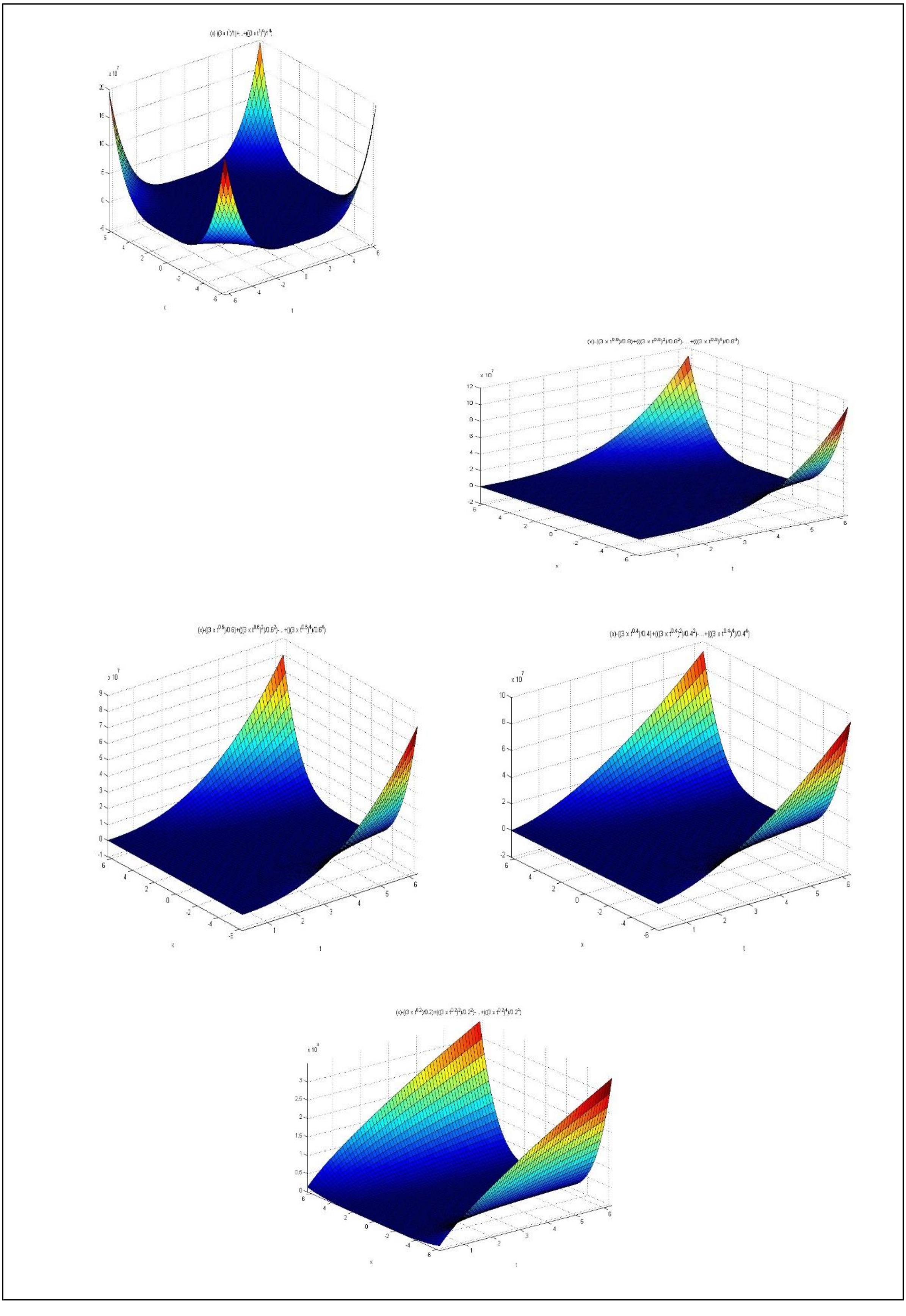

Figure 1. Three-dimensional surface graphs of $u_4(x,t)$ when $0 \le x \le 3, 0 \le t \le 0.1,\ \alpha = 0.2, 0.4, 0.6, 0.8\ , 1$

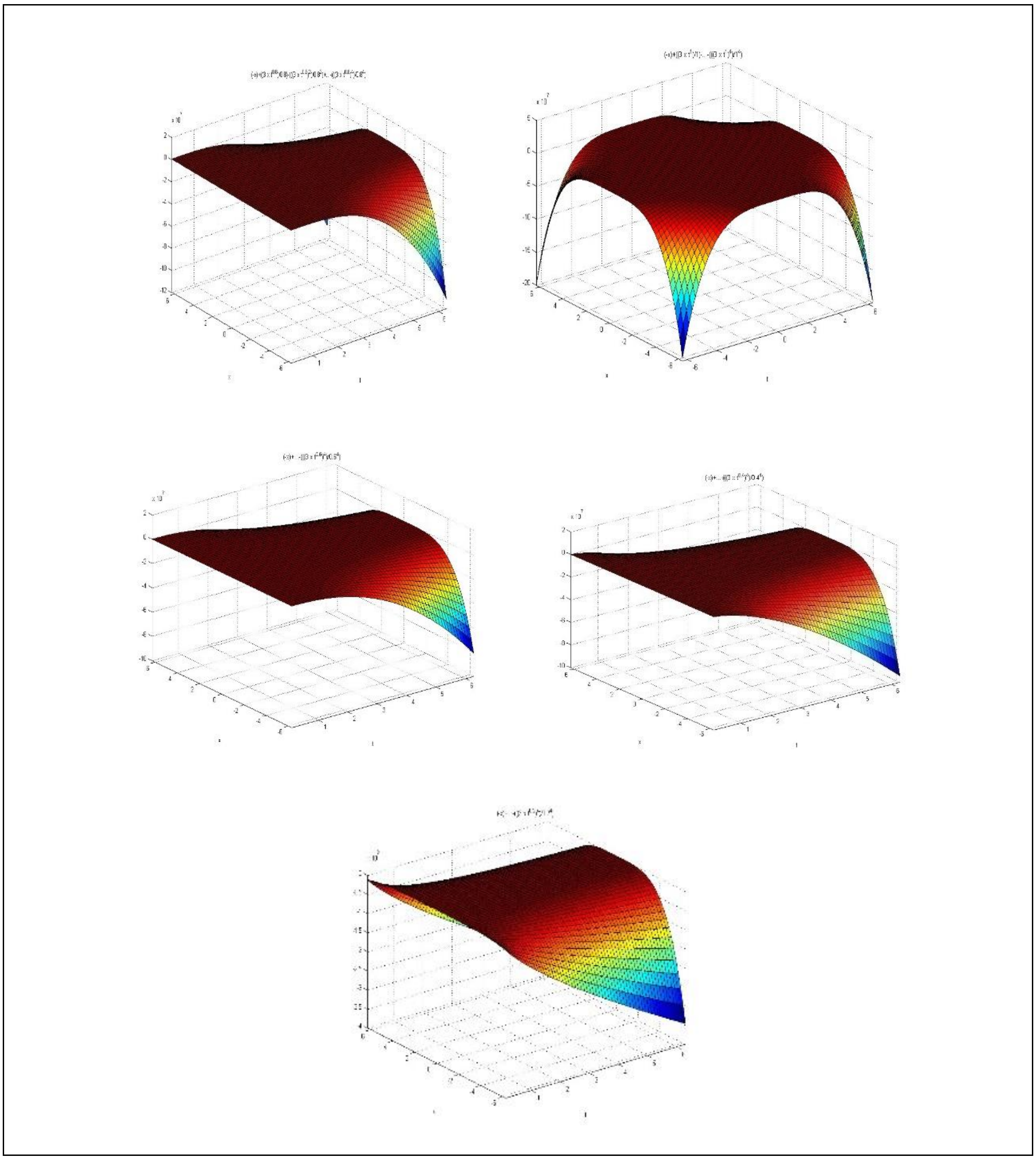

Figure 2. Three-dimensional surface graphs of $v_4(x,t)$ when $0 \le x \le 3$, $0 \le t \le 0.1$, $\alpha = 0.2, 0.4, 0.6, 0.8, 1$

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