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Using the New Integral "Iman Transform" of Bessel's Function to Solve Complementary Applications

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Abstract—In the advance time, Bessel's function plays an important role in many engineering and scientific problems, including equations such as the wave equation, heat equation, Laplace equation, Helmholtz equation, and Schrodinger equation in spherical or cylindrical coordinates. In this paper, author presents Iman transform of Bessel's functions with application for evaluating definite integrals, and exploring its applications and providing examples of problems involving Bessel's functions for integral calculation, thereby minimizing computational efforts.

Keywords—Iman transform; Bessel's function; Numerical applications; Definite integral; transform.

1. Introduction

Recently, integral transforms such as Laplace transform [1,2], Elzaki transform [3,4], Kamal transform [5,6], Mahgoub transform [7], Shehu transform [8], Sawi transform [9], See transform [1], Abaoub Skhem transform [9] ect, have played a crucial role in solving advanced problems in fields such as mathematics, physics, chemistry, social science, biology, nuclear science, and engineering [10]. These transforms are particularly useful for addressing complex challenges like differential equations, and population dynamics.

Similarly, the widespread use of integral transform methods continues to prove valuable in solving various engineering and scientific problems. Bessel's functions are widely applied in solving problems across various fields, including mathematical physics, acoustics, engineering, and sciences like heat transfer, fluid mechanics, vibrations, stress analysis, hydrodynamics, and flux distribution in nuclear reactors, among other.

The purpose of this study is to determine the Iman transform of Bessel's function of the first kind of orders zero, one, and two and to demonstrate the advantages of using the Iman transform in evaluating integrals that involve Bessel's function.

2. Preliminaries

In this section, the definitions of the Iman transform are recalled, along with its fundamental properties.

Definition 1 (Iman transform)

If $f(t) \in F$, $t \geq 0$ then Iman transform of $f(t)$ is defined as [2]:

$$I\{f(t)\} = \frac{1}{v^2}\int_0^\infty e^{-v^2t} f(t)\,\mathrm{dt} = K(v), v > 0 \qquad (1)$$

where I is called the Iman transform operator.

Properties of Iman transform

***(i)* Change of scale**

If $I\{f(t)\} = K(v)$, then $I\{f(at)\} = aK(av)$, where a is arbitrary constant.

Proof: By the definition of Iman transform, we have

$$I\{f(t)\} = \frac{1}{v^2}\int_0^\infty e^{-v^2t} f(t)\,dt, \text{ then}$$

$$I\{f(at)\} = \frac{1}{v^2}\int_0^\infty e^{-v^2t} f(at)\,dt$$

put $at = p \rightarrow adt = dp$ in above equation, we have

$$I\{f(at)\} = \frac{1}{av^2}\int_0^\infty e^{-v^2\frac{p}{a}} f(p)\,dp$$

$$I\{f(at)\} = \left[\frac{1}{a^2\left(\frac{v}{\sqrt{a}}\right)^2}\int_0^\infty e^{-\left(\frac{v}{\sqrt{a}}\right)^2p} f(p)\,dp\right] = \frac{1}{a^2}k\left(\frac{v}{\sqrt{a}}\right)$$

thus, if $I\{f(t)\} = K(v)$, then $I\{f(at)\} = \frac{1}{a^2}k\left(\frac{v}{\sqrt{a}}\right)$

***(ii)* Linearity property**

If $I\{f(t)\} = K(v)$ and $I\{g(t)\} = H(v)$ we have

$$I\{af(t)+bg(t)\} = aI\{f(t)\}+bI\{g(t)\}$$

Then $\quad I\{af(t)+bg(t)\} = aK(v) + bH(v)$

Where a,b are arbitrary constants.

Proof: By the definition of Iman transform, we have

$$I\{f(t)\} = \frac{1}{v^2}\int_0^\infty e^{-v^2t} f(t)\,\mathrm{dt}$$

$$I\{af(t)+bg(t)\} = \frac{1}{v^2}\int_0^\infty e^{-v^2t}[af(t) + bg(t)]\,dt$$

$$= a\left[\frac{1}{v^2}\int_0^\infty e^{-v^2t} f(t)dt\right] + b\left[\frac{1}{v^2}\int_0^\infty e^{-v^2t} g(t)\,dt\right]$$

Then

$$I\{af(t)+bg(t)\} = aI\{f(t)\}+bI\{g(t)\}$$

and

$$I\{af(t)+bg(t)\} = aK(v) + bH(v)$$

Where a,b are arbitrary constants.

Definition 2: Iman transform of the derivatives of the function *f(t)* [3-4]

Let $K(v)$ be Iman transform of [$I\{f(t)\} = K(v)$], then

a) $I\left\{\frac{df(t)}{dt}\right\} = v^2\, K(v) \; - \frac{1}{v^2} f(0)$

b) $I\left\{\frac{d^2f(t)}{dt^2}\right\} = v^4\, K(v) \; - \; f(0) - \frac{1}{v^2} f'(0)$

c) $I\left\{\frac{df(t)}{dt}\right\} = v^{2n}\, K(v) \; - \sum_{k=0}^{n=1} \frac{1}{v^4 - 2n + 2k} f^{(k)}(0)$

Definition 3: Convolution of two functions

The convolution of two functions $f(t)$ and $g(t)$ is represented as $f(t) * g(t)$ and is defined by:

$$f(t) * g(t) = f * g = \int_0^t f(t)\, g(x - t)\, dt$$

$$= \int_0^t g(t)\, f(x - t)\, dt$$

Theorem 1: Convolution theorem for Iman transforms

If $I\{f(t)\} = K_1(v)$ and $I\{g(t)\} = K_2(v)$, then

$I\{f(t) * g(t)\} = v^2 I\{f(t)\} * I\{g(t)\} = v^2 K_1(v) \,.\, K_2(v)$

Proof: By the definition of Iman transform, we have

$$I\{f(t)\} = \frac{1}{v^2}\int_0^\infty e^{-v^2 t}\, f(t)\, \mathrm{dt}$$

$$I\{f(t) * g(t)\} = \frac{1}{v^2}\int_0^\infty e^{-v^2 t}\, [\, f(t) * g(t)\,]\mathrm{dt}$$

From the definition of convolution of two functions, we obtain

$$I\{f(t) * g(t)\} = \frac{1}{v^2}\int_0^\infty e^{-v^2 t}\, [\, \int_0^t f(x)\, g(t - x)\,]dt$$

Changing the order of integration gives

$$I\{f(t) * g(t)\} = \frac{1}{v^2}\int_0^\infty f(x)\; [\, \int_x^\infty e^{-v^2 t}\, g(t - x) dt\,]dx$$

put $t - x = u$ so that $dt = du$ in above equation, we have

$$I\{f(t) * g(t)\} = \frac{1}{v^2}\int_0^\infty f(x)\; [\, \int_x^\infty e^{-v^2(u+x)}\, g(u) du\,]dx$$

$$I\{f(t) * g(t)\} = \frac{1}{v^2}\int_0^\infty f(x) e^{-v^2 x}\, [\, \int_0^\infty e^{-v^2 u}\, g(u) du\,]dx$$

$$I\{f(t) * g(t)\} = \int_0^\infty f(x) e^{-v^2 x}\, [\, I\, \{g(t)\}] dx$$

$$I\{f(t) * g(t)\} = v^2 I\{g(t)\} [\frac{1}{v^2}\int_0^\infty e^{-v^2 x}\, f(x)\, dx]$$

$$I\{f(t) * g(t)\} = v^2 I\{f(t)\} * I\{g(t)\} = v^2 K_1(v) \,.\, K_2(v)$$

Table .1. Iman transform of some elementary functions [2]

S.N	Function, $f(t)$	$I\{f(\mathrm{t})\} = K(v)$
1	1	$\frac{1}{v^4}$
2	t	$\frac{1}{v^6}$
3	$t^n, n \in N$	$\frac{n!}{v^{2n+4}}$
4	$t^n, n > -1$	$\frac{\Gamma(n+1)}{v^{2n+4}}$
5	e^{at}	$\frac{1}{v^{4-av^2}}$
6	e^{-at}	$\frac{1}{v^{4+av^2}}$
7	$sin\,(at)$	$\frac{a}{v^2(v^{4+a^2})}$
8	$cos\,(at)$	$\frac{1}{v^{4+a^2}}$
9	$sinh\,(at)$	$\frac{a}{v^2(v^{4-a^2})}$
10	$cosh\,(at)$	$\frac{1}{v^{4-a^2}}$

Definition 4: Inverse Iman transform

If $I\{f(t)\} = K(v)$ then $f(t)$ is called the inverse Iman transform of $K(v)$ and mathematically it is denoted by

$$f(t) = I^{-1}\{K(v)\}$$

Where the operator I^{-1} is the inverse of Iman transform.

Table .2. Inverse Iman transform of Some elementary functions [4, 5]

S.N	$K(v)$	$f(t) = I^{-1}\{K(v)\}$
1	$\frac{1}{v^4}$	1
2	$\frac{1}{v^6}$	t
3	$\frac{1}{v^{2n+4}}$, n	$n!\, t^n$
4	$\frac{1}{v^{2n+4}}$, $n > -1$	$\Gamma(n+1)\ \ t^n, n > -1$
5	$\frac{1}{v^4 - av^2}$	e^{at}
6	$\frac{1}{v^4 + av^2}$	e^{-at}
7	$\frac{1}{v^2(v^4 + a^2)}$	$\frac{\sin(at)}{a}$
8	$\frac{1}{(v^4 + a^2)}$	$\cos(at)$
9	$\frac{1}{v^2(v^4 - a^2)}$	$\frac{\sinh(at)}{a}$
10	$\frac{1}{(v^4 - a^2)}$	$\cosh(at)$

Theorem 2: Bessel's functions of first kind different order [8-9]

Bessel's function of order *n*, where *n* ∈ N is given by $J_n(t)$

$$= \frac{t^n}{2^n n!}\left[1 - \frac{t^2}{2.(2n+2)} + \frac{t^4}{2.4(2n+2)(2n+4)} - \frac{t^6}{2.4.6.(2n+2)(2n+4)(2n+6)} + \cdots\right] \quad (2)$$

Where n = 0, we have Bessel's function of zero order and it denoted by $J_0(t)$ and it is given by

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 4^2} - \frac{t^6}{2^2 4^2 6^2} + \cdots \quad (3)$$

And when *n =1*, we have Bessel's function of order one, it is given by

$$J_1(t) = \frac{t}{2} - \frac{t^3}{2^3 \cdot 4} + \frac{t^5}{2^2 . 4^2 .6} - \frac{t^7}{2^2 .\ 4^2 .\ 6^2 .8} + \cdots \quad (4)$$

Another form of equation (3) can be written as

$$J_1(t) = \frac{t}{2} - \frac{t^3}{2^3 \cdot\ 2!} + \frac{t^5}{2^5 .\ 2! .\ 3!} - \frac{t^7}{2^7 .\ 3! .4!} + \cdots \quad (5)$$

For *n =2*, Bessel's function of order two, it is given by

$$J_2(t) = \frac{t^2}{2.4} - \frac{t^4}{2^2 .4.6} + \frac{t^6}{2^2 . 4^2 .6.8} - \frac{t^8}{2^2 .\ 4^2 .\ 6^2 .8.10} + \cdots \quad (6)$$

Definition 5: Relation between $J_0(t)$ and $J_1(t)$ [8]

$$\frac{d}{dt}[J_0(t)] = -J_1(t) \tag{7}$$

Definition 6: Relation between $J_0(t)$ and $J_2(t)$ [9]

$$J_2(t) = J_0(t) + 2J''_0(t) \tag{8}$$

Theorem 3: Iman transform of Bessel's functions of first kind

***(i)* Iman transform of $J_0(t)$**

Applying Iman transform of equation (3), both sides, we have:

$$I[J_0(t)] = I[1] - \frac{1}{2^2} I[t^2] + \frac{1}{2^2.4^2} I[t^4] - \frac{1}{2^2.\ 4^2.\ 6^2} I[t^6] + \cdots$$

$$= \frac{1}{v^4} - \frac{1}{2^2}\frac{2!}{.\ v^8} + \frac{1}{2^2.\ 4^2.\ v^{12}}\frac{4!}{} - \frac{1}{2^2.\ 4^2.\ 6^2}\frac{6!}{v^{16}} + \cdots$$

$$= \frac{1}{v^4}\left[1 - \frac{1}{2^2}\frac{2!}{.\ v^4} + \frac{1}{2^2.\ 4^2.}\frac{4!}{v^8} - \frac{1}{2^2.\ 4^2.\ 6^2}\frac{6!}{v^{12}} + \cdots\right]$$

$$= \frac{1}{v^4}\left[1 - \frac{1}{2}\left(\frac{1}{v^4}\right) + \frac{3.1}{2.4}\left(\frac{1}{v^4}\right)^2 - \frac{5.3.1}{2.4}\left(\frac{1}{v^4}\right)^3 + \cdots\right]$$

$$= \frac{1}{v^4}\left(1 + \frac{1}{v^4}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{v^2\sqrt{1+v^4}} \tag{9}$$

***(ii)* Iman transform of $J_1(t)$**

From (7), we have $\frac{d}{dt}[J_0(t)] = -J_1(t)$ then

$$I[J_1(t)] = -I\frac{d}{dt}[J_0(t)]$$

$$= -\left[\frac{-1}{v^2}J_0(0) + v^2 I[J_0(t)]\right]$$

$$= \frac{1}{v^2}J_0(0) - v^2 I[J_0(t)]]$$

$$= \frac{1}{v^2}(1) - \frac{v^2}{v^4\sqrt{1+\frac{1}{v^4}}}$$

$$= \frac{1}{v^2}\left[1 - \frac{v^2}{\sqrt{1+v^4}}\right] \tag{10}$$

***(iii)* Iman transform of $J_2(t)$**

Taking Iman transform of equation (8), both sides, we get

$$I[J_2(t)] = I[J_0(t)] + 2I[J''_0(t)]$$

$$= \frac{1}{v^2\sqrt{1+v^4}} + 2[v^4\, I[J_0(t)] - \frac{1}{v^2} J'_0(0) + J_0(0)]$$

$$= \frac{1}{v^2\sqrt{1+v^4}} + 2[\frac{v^2}{v^2\sqrt{1+v^4}} - 0 - 1]$$

$$= \frac{1}{v^2\sqrt{1+v^4}} + \frac{2}{\sqrt{1+v^4}} - 2$$

$$= \frac{1+2v^2 - 2v^2\sqrt{1+v^4}}{v^2\sqrt{1+v^4}} \qquad (11)$$

***(iv)* Iman transform of $J_0(at)$**

Since $I\,[J_0(t)] = \frac{1}{v^2\sqrt{1+v^4}}$

Now, using change of scale property of Iman transform, we have:

$$I[J_0(at)] = \frac{1}{a^2}\left[\frac{1}{(\frac{v}{\sqrt{a}})^2\sqrt{1+(\frac{v}{\sqrt{a}})^4}}\right]$$

$$= \frac{1}{v^2}\left[\frac{1}{\sqrt{a^2+v^4}}\right] \qquad (12)$$

***(v)* Iman transform of $J_1(at)$**

Since $I[J_1(t)] = \frac{1}{v^2}[1 - \frac{v^2}{\sqrt{1+v^4}}]$

Then $I[J_2(t)] = \frac{1}{a^2}\left[\frac{1}{(\frac{v}{\sqrt{a}})^2}[1 - \frac{(\frac{v}{\sqrt{a}})^2}{\sqrt{1+(\frac{v}{\sqrt{a}})^4}}]\right]$

$$= \left[\frac{1}{av^2} - \frac{1}{a^3\sqrt{a^2+v^4}}\right] \qquad (13)$$

***(vi)* Iman transform of $J_2(at)$**

Since $I[J_2(t)] = \frac{1+2v^2 - 2v^2\sqrt{1+v^4}}{v^2\sqrt{1+v^4}}$

Then $I[J_2(at)] = \frac{1}{a^2}\left[\frac{1+2\frac{v^2}{a^2} - 2\frac{v^2}{a^2}\sqrt{a^2+v^4}}{\frac{v^2}{a^2}\sqrt{a^2+v^4}}\right]$

$$= \frac{1}{a^2v^2}\left[\frac{a^2+2av^2 - 2v^2\sqrt{a^2+v^4}}{\sqrt{a^2+v^4}}\right] \qquad (14)$$

Theorem 4: First shifting Theorem for Iman transform

Let $I\,[f(t)] = K(v)$ and $a \in R$,then

$I\,[\,e^{at}f(t)] = (\,1 - \frac{a}{v^2})k\,(\,v\sqrt{1 - \frac{a}{v^2}}\,)$

Proof: From definition of a new integral transform, we have

$$I\,[\,e^{at}f(t)] \;=\; \frac{1}{v^2}\int_0^{\infty} e^{at}\;f(t)\,e^{-v^2 t}$$

$$= \;\frac{1}{v^2}\int_0^{\infty} f(t)\,e^{-[v^2-\,a]t}\;dt$$

$$= \;(\,1\;-\;\frac{a}{v^2})\frac{1}{(\,v\sqrt{1-\frac{a}{v^2}}\,)^2}\int_0^{\infty} f(t)\,e^{-(\,v\sqrt{1-\frac{a}{v^2}}\,)^2 t}\;dt$$

$$= (\,1 - \frac{a}{v^2})k\,(\,v\sqrt{1 - \frac{a}{v^2}}\,) \tag{15}$$

3. Numerical Applications

Application 1: Evaluate the integral

$$E(t) \;=\; \int_0^t J_0\,(u)\,J_0(t-\,u)du \tag{16}$$

Taking Iman transform to both sides of equation (16), we have

$$I\{E(t)\} \;=\; I\{\int_0^t J_0\,(u)\,J_0(t-\,u)du\,\} \tag{17}$$

Applying convolution theorem of Iman transform on (17), we have

$$I\{E(t)\} \;= v^2\,I\{\,J_0(t)\}\,I\{\,J_0(t)\}$$

$$= v^2[\;\frac{1}{v^2\sqrt{1+\,v^4}}\,][\;\frac{1}{v^2\sqrt{1+\,v^4}}]$$

$$= \frac{1}{v^{2(1+v^4)}} \tag{18}$$

Now operating inverse Iman transform on both side on (18), we have

$$E(t) = \;\; I^{-1}\{\;\frac{1}{v^{2(1+v^4)}}\;\} = \sin(t) \tag{19}$$

Which is the exact solution needed for (16).

Application 2: Evaluate the integral

$$E(t) \;=\; \int_0^t J_0\,(u)\,J_1(t-\,u)du \tag{20}$$

Taking Iman transform to both sides of equation (20), we have

$$I\{E(t)\} \;=\; I\{\int_0^t J_0\,(u)\,J_1(t-\,u)du\,\} \tag{21}$$

Using convolution theorem of Iman transform on (21), we have

$$I\{E(t)\} \;= v^2\,I\{\,J_0(t)\}\,I\{\,J_1(t)\}$$

$$= v^2[\frac{1}{v^2\sqrt{1+v^4}}][\frac{1}{v^2}][1-\frac{v^2}{\sqrt{1+v^4}}]$$

$$= \frac{1}{v^2\sqrt{(1+v^4)}} - \frac{1}{1+v^4} \tag{22}$$

Take inverse to both sides on (22), we get

$$E(t) = I^{-1}\left\{\frac{1}{v^2\sqrt{(1+v^4)}}\right\} - I^{-1}\left\{\frac{1}{(1+v^4)}\right\}$$

$$= J_0(t) - cos\,(t) \tag{23}$$

The exact solution of (20) that is required.

Example 1. Evaluate the integral

$$E(t) = \int_0^t J_1(t-u)du \tag{24}$$

Applying the Iman transform on both sides of equation (24), we have

$$I\{E(t)\} = I\left\{\int_0^t J_1(t-u)du\right\} \tag{25}$$

Using convolution theorem of Iman transform on (25), we have

$$I\{E(t)\} = v^2\, I\{1\}\, _-\, I\{J_1(t)\}$$

$$= v^2[\frac{1}{v^4}][\frac{1}{v^2}[1-\frac{v^2}{\sqrt{1+v^4}}]$$

$$= \frac{1}{v^4} - \frac{1}{v^2\sqrt{1+v^4}}] \tag{26}$$

Now taking inverse Iman transform on both sides on (26), we have

$$E(t) = I^{-1}\left\{\frac{1}{v^4}\right\} - I^{-1}\left\{\frac{1}{v^2\sqrt{(1+v^4)}}\right\}$$

$$= 1 - J_0(t) \tag{27}$$

Which is the required solution of equation (24).

Conclusion

In this work, the Iman transform is employed to solve Bessel functions of the first kind of orders zero, one, and two, denoted as $J_0(t)$, $J_1(t)$, $J_2(t)$, $J_0(at)$, $J_1(at)$, and $J_2(at)$. Additionally, they obtained the Iman transform for expressions involving these functions, such as $e^{kt}J_0(t)$, $e^{kt}J_1(t)$, $e^{kt}J_2(t)$, using the translation and scaling properties of the Iman transform. These findings are significant for the evaluation of improper integrals that include Bessel functions in the integrand. Additionally, we explore several properties and theorems related to the Iman transform, we applied this transform to three different examples. The results of this study may also be useful in future research for solving Bessel's equations.

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