Derna University Journal of Applied Sciences

Vol. 3, Issue 1, 2026

ISSN: 3079-0719 (Online) | DOI: https://doi.org/10.58987/6m7qh323

Structures and $*$-Embeddability of Commutative Semigroups with Involution of Order 3, 4, and 5

Gbrel Albkwre*[1], Hassan Al Hwaeer[2], Cheng-Han Pan[3]

[1] Department of Mathematics, Faculty of Science, University of Derna, Libya
[2] Department of Mathematics and Computer Applications, Al-Nahrain University, Baghdad, Iraq
[3] Independent Researcher, Taitung 950009, Taiwan (R.O.C.)

* gbrel.albkwre@uod.edu.ly

Received: May 11, 2026; **Accepted**: Jun 27, 2026.

Abstract—This paper investigates the $*$-embeddability of proper $*$-semigroups into rings with proper involution. We establish a criterion characterizing $*$-embeddability in terms of proper $*$-ideals of the semigroup ring $\mathbb{Z}[S]$. In particular, if the equation $XX^*= 0$ has only the trivial solution in $\mathbb{Z}[S]$, then $(S,*)$ is $*$-embeddable in $\mathbb{Z}[S]$. We also give examples illustrating both $*$-embeddability and the failure of $*$-embeddability. Finally, we computationally investigate all commutative proper $*$-semigroups of orders 3, 4, and 5, consisting of 5072 cases in total. Computation shows that 4940 cases have only the trivial solution to $XX^* = 0$, while 132 cases admit nontrivial solutions. In every nontrivial case, the computation finds a solution with coefficients in $\{-1,0,1\}$ whose sum is 0, suggesting a general pattern for solutions of $XX^* = 0$.

Keywords—Semigroup; $*$-Semigroup ring; $*$-Embeddability; Involution

1. Introduction

One of the important classes of rings is the class of semigroup rings. In this investigation, we focus on a special class of semigroup $*$-rings induced by a proper $*$-semigroup. In particular, we are interested in the following general question: does there exist a ring R with a proper involution in which S is $*$-embeddable?

Shehadah [1] has proved that every inverse semigroup is $*$-embeddable into a ring with proper involution when its involution is the inverse mapping. More recently, Lee [2] proved that a finite $*$-semigroup is embeddable into a semigroup of complex matrices with conjugate transposition if and only if it is an inverse semigroup. On the other hand, Shehadah [3] showed that a general proper $*$-semigroup need not be $*$-embeddable into any ring with proper involution. In particular, the natural semigroup ring $(\mathbb{Z}[S],*)$ need not be a proper $*$-ring, as witnessed by nonzero elements $A, B \in \mathbb{Z}[S]$ satisfying $AA^* = 0, BB^* = 0$. For more research related to the question, we refer the reader to [4-7].

In this paper, we computationally investigate all commutative proper $*$-semigroups $(S,*)$ of orders 3, 4, and 5, comprising 5072 cases in total. For each case, we determine whether the equation $XX^* = 0$ has only the trivial solution in $\mathbb{Z}[S]$. Whenever a nontrivial solution exists, our exact search finds one whose coefficients belong to $\{-1,0,1\}$ and sum to 0. These computations support the conjecture that, whenever $XX^* = 0$ admits a nonzero solution in $\mathbb{Z}[S]$, it also admits a nontrivial solution with coefficients in $\{-1,0,1\}$ summing to 0, although a proof has not yet been achieved.

To carry out this investigation, we used two Groups, Algorithms, and Programming (GAP) programs [8]. The first enumerated all commutative semigroups of orders 3, 4 and 5 with proper involution $*$. The second used these semigroups to analyze the solutions of $XX^* = 0$ in $\mathbb{Z}[S]$, applying sum-of-squares, modular, and exact searches to the resulting homogeneous systems.

2. P $*$-Semigroup and P $*$-Ring

Since our main goal in this paper is to answer the mentioned question for $*$-semigroups and $*$-rings with proper involution, we need some more definitions and facts. For more details, we refer the reader to [9-11].

Definition 2.1

Given any semigroup S, a map $*: S \rightarrow S$ is called an *involution* of S if

(1) $(a^*)^* = a \quad \forall a \in S$;

(2) $(ab)^* = b^*a^* \; \forall a, b \in S$.

$(S,*)$ is called a $*$-semigroup or semigroup with involution $*$.

Definition 2.2

The involution $*$ is called proper if $aa^* = ab^* = bb^*$ implies that $a = b \; \forall a, b \in S$. A commutative semigroup with a proper involution $*$ is called a *P $*$-semigroup*.

Definition 2.3

Let R be a ring. An *involution* on R is a map $*: R \rightarrow R$ such that

(1) $(A + B)^* = A^* + B^*$;

(2) $(AB)^* = B^*A^*$;

(3) $(A^*)^* = A \; \forall A, B \in R$.

Definition 2.4

An involution $*$ on a ring R is called a *proper involution* if for every $A \in R$, $AA^* = 0$ implies $A = 0$. In this case $(R,*)$ is called a *P $*$-ring*. This is consistent with the definition of proper involution on semigroups.

An example of a P $*$-ring is $(\mathbb{C},*)$, where $\mathbb{C}$ is the complex field and $*$ is the conjugate operator. Also, $(M_{n\times n}(\mathbb{C}),*)$ is a P $*$-ring, where $*$ is the Hermitian operator on the ring of all $n \times n$ matrices over $\mathbb{C}$.

Definition 2.5

Let S be a semigroup and let R be a ring. We say that S is *embedded* in R if there is an injective map $f : S \rightarrow R$ such that $f(ab) = f(a)f(b) \; \forall a, b \in S$. *An* $(S,*)$ is called $*$-embedded in the $*$-ring $(R,*)$ if the map f also satisfies $f(a^*) = \left(f(a)\right)^* \; \forall a \in S$.

Definition 2.6

Let $(S,*)$ be a $*$-semigroup and let $(R,*)$ be a $*$-ring. We define the *semigroup* ring as follows:

$$R[S] = \left\{ \sum_{i=1}^{n} a_i g_i : a_i \in R, g_i \in S, n \in \mathbb{Z}^+ \right\},$$

where $\sum_{i=1}^{n} a_i g_i$ is just a formal symbol. Thus, $\sum a_i g_i = \sum b_i g_i \Leftrightarrow a_i = b_i \quad \forall i = 1,2,\ldots,n.$

Addition and multiplication are defined as follows:

$$\sum_{i=1}^{n} a_i g_i + \sum_{i=1}^{n} b_i g_i = \sum_{i=1}^{n} (a_i + b_i) g_i$$

$$\left(\sum_{i=1}^{n} a_i g_i \right) \left(\sum_{i=1}^{n} b_i g_i \right) = \sum_{k=1}^{n} c_k g_k \,,$$

where $c_k = \sum_{i=1}^{n} a_i b_i$, which is the sum taken over all pairs (a_i, b_j) such that $g_i g_j = g_k$. Define the involution on $R[S]$ as the following:

$$\left(\sum a_i g_i \right)^* = \sum a_i^* g_i^* \quad \forall a_i \in R, g_i \in S.$$

3. P $*$-Ideal

Given a proper $*$-semigroup $(S,*)$, the problem is to find a proper $*$-ring $(R, *)$ and a $*$-embedding function $f : (S,*) \to (R,*)$.

Let $(S,*)$ be a proper $*$-semigroup. The natural extension of $*$ to the semigroup ring $(\mathbb{Z}[S],*)$ of $(\mathbb{Z}, identity)$ over $(S,*)$ may not be a P $*$-ring.

Definition 3.1.

Let $(R,*)$ be a $*$-ring. An ideal $\mathfrak{a}$ of R is said to be a $*$-*ideal* if it satisfies $\mathfrak{a}^* = \mathfrak{a}$ i.e., $a^* \in \mathfrak{a}$ if and only if $a \in \mathfrak{a}$.

We say that a $*$-ideal $\mathfrak{a}$ is proper if for any element $a \in R$, $aa^* \in \mathfrak{a}$ implies that $a \in \mathfrak{a}$. In particular, we have the following simple observation that we will leave without proof.

Proposition 3.2

Let $\mathfrak{a}$ be an ideal of a $*$-ring R. Then

(1) $\mathfrak{a}$ is a $*$-ideal if and only if $R/\mathfrak{a}$ is a $*$-ring with induced involution. Moreover,
(2) $\mathfrak{a}$ is proper if and only if $R/\mathfrak{a}$ is proper.

Theorem 3.3

The following conditions are equivalent for a proper $*$-semigroup S.

(1) S is $*$-embeddable into some proper $*$-ring.
(2) There is a proper $*$-ideal $\mathfrak{a}$ of $\mathbb{Z}[S]$ satisfying $\mathfrak{a} \cap (S - S) = \{0\}$.

Proof

We can easily check that for an ideal $\mathfrak{a}$ of $\mathbb{Z}[S]$, the composition $S \to \mathbb{Z}[S] \to \mathbb{Z}[S]/\mathfrak{a}$ is injective if and only if $\mathfrak{a} \cap (S - S) = \{0\}$. Assume there is a $*$-embedding $f : S \hookrightarrow R$ into a proper $*$-ring R. By Proposition 3.2, f factors as

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "i"'] & R \\ \mathbb{Z}[S] \arrow[ur, "g"'] & \end{tikzcd}$$

$\mathfrak{a} :=$ Ker g. Since g is a $*$-homomorphism, $\mathfrak{a}$ is a $*$-ideal. Since $\mathbb{Z}[S]/\mathfrak{a}$ is a $*$-subring of a proper $*$-ring R, it is also a proper $*$-ring. This shows that $\mathfrak{a}$ is a proper $*$-ideal. Thus, we need to check $\mathfrak{a} \cap (S - S) = \{0\}$. Let $s, t \in S$ such that $s - t \in \mathfrak{a}$. Then $f(s) = g(s) = g(t) = f(t)$. Therefore, we get $s = t$.

Conversely, suppose there is a proper $*$-ideal $\mathfrak{a}$ of $\mathbb{Z}[S]$ satisfying $\mathfrak{a} \cap (S - S) = \{0\}$. Then $\mathbb{Z}[S]/\mathfrak{a}$ is a proper $*$-ring and the canonical homomorphism $S \to \mathbb{Z}[S]/\mathfrak{a}$ is injective. □

Corollary 3.4

Let $(S,*)$ be a proper $*$-semigroup, let $A_1, A_2 \in \mathbb{Z}[S]$ be such that $A_1A_1^* = 0 = A_2A_2^*$, and $C \in \mathbb{Z}[S]$ be such that $CC^* = m_1A_1 + m_2A_2$ for some integers m_1 and m_2. If $D = s_1 - s_2$ is a linear combination of $A_1, A_2, C \in \mathbb{Z}[S]$, then there is no P $*$-ring in which S is $*$-embeddable.

Proof

Let $\mathfrak{a}$ be a proper $*$-ideal of $\mathbb{Z}[S]$. Since $A_1A_1^* = A_2A_2^* = 0 \in \mathfrak{a}$, we have $A_1, A_2 \in \mathfrak{a}$ as $\mathfrak{a}$ is a proper $*$-ideal. Therefore, $CC^* = m_1A_1 + m_2A_2 \in \mathfrak{a}$ and again using the properness of $\mathfrak{a}$, we obtain $C \in \mathfrak{a}$. Now we conclude that $D = s_1 - s_2 \in \mathfrak{a} \cap (S - S) \neq \{0\}$. Thus, Theorem 3.3 shows that there is no P $*$-ring in which S is $*$-embeddable.

We have the following simple corollary, which we leave without proof. □

Corollary 3.5

If $XX^* = 0$ has only trivial solution for every $X \ in\ \mathbb{Z}[S]$, then $\mathbb{Z}[S]$ is itself a proper $*$-ring in which S is $*$-embeddable.

4. A Counterexample on $*$-Embeddability

The following important example shows that it is not always possible, given a semigroup S with proper involution $*$, to find a ring R with proper involution $*$ for which S is $*$-embeddable.

Let $S \subseteq \mathbb{Z}^3$ be such that

$$S = \{s_1 = (1,1,1), s_2 = (-1,1,1), s_3 = (-1,-1,1), s_4 = (1,-1,1)\}.$$

Clearly, S is a semigroup under pointwise multiplication:

$$(a_1, b_1, c_1)(a_2, b_2, c_2) = (a_1a_2, b_1b_2, c_1c_2)$$

Define a map $*\colon S \to S$ by $(a,b,c)^* = (a,ab,c)$. This map satisfies $(a,b,c)^{**} = (a,b,c)$ and

$$\begin{aligned}[(a_1,b_1,c_1)(a_2,b_2,c_2)]^* &= (a_1a_2,b_1b_2,c_1c_2)^* \\ &= (a_1a_2,a_1a_2b_1b_2,c_1c_2) \\ &= (a_2,b_2,c_2)^*(a_1,b_1,c_1)^* \quad \forall (a_1,b_1,c_1),(a_2,b_2,c_2) \in S.\end{aligned}$$

Thus, $*$ is an involution on S. To show the properness of $*$, let $s_1 = (a_1,b_1,c_1), s_2 = (a_2,b_2,c_2) \in S$ be such that $s_1s_1^* = s_1s_2^* = s_2s_2^*$. Then

$$(a_1,b_1,c_1)(a_1,a_1b_1,c_1) = (a_1,b_1,c_1)(a_2,a_2b_2,c_2) = (a_2,b_2,c_2)(a_2,a_2b_2,c_2)$$

$$(a_1^2,a_1b_1^2,c_1^2) = (a_1a_2,a_2b_1b_2,c_1c_2) = (a_2^2,a_2b_2^2,c_2^2).$$

Thus, we have

$$a_1^2 = a_1a_2 = a_2^2 \tag{1}$$

$$a_1b_1^2 = a_2b_1b_2 = a_2b_2^2 \tag{2}$$

$$c_1^2 = c_1c_2 = c_2^2 \tag{3}$$

Since $a_1,a_2,b_1,b_2,c_1,c_2 \neq 0$, from Eq. (1) and Eq. (3) we get $a_1 = a_2,\ c_1 = c_2$. Substituting into Eq. (2) we get $b_1^2 = b_1b_2 = b_2^2$. This implies $b_1 = b_2$. So $s_1 = s_2$ and hence $*$ is proper.

Table 1 shows the multiplication of S and the column under $*$ specifies the involution on S.

Table 1. Multiplication table of S with the values of the involution $*$ in the rightmost column

$\cdot$	s_1	s_2	s_3	s_4	$*$
s_1	s_1	s_2	s_3	s_4	s_1
s_2	s_2	s_1	s_4	s_3	s_3
s_3	s_3	s_4	s_1	s_2	s_2
s_4	s_4	s_3	s_2	s_1	s_4

Let $(\mathbb{Z}[S],*)$ be the $*$-semigroup ring of ($\mathbb{Z}$, *identity)* over $(S,*)$.

Let $X = \sum_{i=1}^{4} x_is_i$ be such that $XX^* = 0$

$$XX^* = (x_1s_1 + x_2s_2 + x_3s_3 + x_4s_4)(x_1s_1^* + x_2s_2^* + x_3s_3^* + x_4s_4^*) = 0$$

$$x_1^2s_1 + x_1x_2s_3 + x_1x_3s_2 + x_1x_4s_4$$

$$+x_2x_1s_2 + x_2^2s_4 + x_2x_3s_1 + x_2x_4s_3$$

$$+x_3x_1s_3 + x_3x_2s_1 + x_3^2s_4 + x_3x_4s_2$$

$$+x_4x_1s_4 + x_4x_2s_2 + x_4x_3s_3 + x_4^2s_1 = 0$$

$$s_1(x_1^2 + x_2x_3 + x_3x_2 + x_4^2) + s_2(x_1x_3 + x_2x_1 + x_3x_4 + x_4x_2)$$

$$+s_3(x_1x_2 + x_2x_4 + x_3x_1 + x_4x_3) + s_4(x_1x_4 + x_2^2 + x_3^2 + x_4x_1) = 0$$

Now, let

$$f_1 = x_1^2 + 2x_3x_2 + x_4^2 = 0$$
$$f_2 = x_1x_3 + x_1x_2 + x_3x_4 + x_2x_4 = 0$$
$$f_3 = x_1x_3 + x_1x_2 + x_3x_4 + x_2x_4 = 0$$
$$f_4 = x_2^2 + 2x_1x_4 + x_3^2 = 0$$

Notice that $f_2 = f_3$ and

$$f_1 + f_4 = (x_1 + x_4)^2 + (x_2 + x_3)^2 = 0$$
$$f_2 = (x_1 + x_4)(x_2 + x_3) = 0$$

So, we get $x_1 = -x_4$, $x_2 = -x_3$, and $x_2 = \pm x_1$

$$X = t_1s_1 + t_1s_2 - t_1s_3 - t_1s_4$$

or

$$X = t_1s_1 - t_1s_2 + t_1s_3 - t_1s_4, \qquad t_1 \in \mathbb{Z}.$$

Thus, the solution of the equation $XX^* = 0$ is nontrivial in this case.

Let

$$A = s_1 + s_2 - s_3 - s_4$$
$$B = s_1 - s_2 + s_3 - s_4$$
$$C = 2s_1 + 3s_2 - 3s_3 - 2s_4$$

Then,

$$\begin{aligned}AA^* &= (s_1 + s_2 - s_3 - s_4)(s_1 - s_2 + s_3 - s_4)\\ &= (s_1 - s_2 + s_3 - s_4) + (s_2 - s_1 + s_4 - s_3) + (-s_3 + s_4 - s_1 + s_2)\\ &\qquad +(-s_4 + s_3 - s_2 + s_1) = 0\\ BB^* &= (s_1 - s_2 + s_3 - s_4)(s_1 + s_2 - s_3 - s_4) = 0\\ CC^* &= (2s_1 + 3s_2 - 3s_3 - 2s_4)(2s_1 - 3s_2 + 3s_3 - 2s_4)\\ &= (4s_1 - 6s_2 + 6s_3 - 4s_4) + (6s_2 - 9s_1 + 9s_4 - 6s_3)\\ &\quad +(-6s_3 + 9s_4 - 9s_1 + 6s_2) + (-4s_4 + 6s_3 - 6s_2 + 4s_1)\\ &= -10s_1 + 10s_4\\ &= -5A - 5B.\end{aligned}$$

Let $D = C - 2A$, then $D = s_2 - s_3$. By Corollary 3.4, $(S,*)$ is not $*$-embeddable.

5. An Example of $*$-Embeddability

Let $S \subseteq \mathbb{Z}^2$ be such that

$$S = \{s_1 = (1,1), s_2 = (0,1), s_3 = (-1,1), s_4 = (0,-1)\}.$$

It is clear that S is a semigroup under pointwise multiplication $(a,b)(c,d) = (ac,bd)$.

Define a map $*: S \to S$ by $(a,b)^* = (ab,b)$. This map satisfies $(a,b)^{**} = (a,b)$ and $\big((a,b)(c,d)\big)^* = (ac,bd)^* = (abcd,bd) = (c,d)^*(a,b)^*$. So $*$ is an involution on S.

We now verify that $*$ is proper. Let $s_1 = (a,b), s_2 = (c,d) \in S$ be such that $s_1 s_1^* = s_1 s_2^* = s_2 s_2^*$. Then

$$(a,b)(ab,b) = (a,b)(cd,d) = (c,d)(cd,d)$$
$$(a^2b,b^2) = (acd,bd) = (c^2d,d^2)$$
$$a^2b = acd = c^2d, b^2 = bd = d^2.$$

Since b,d are nonzero, we have $a = c$ and $b = d$. Then, $(a,b) = (c,d)$, so $s_1 = s_2$. Hence $*$ is proper.

The following Table 2 shows the multiplication on S and the column under $*$ specifies the involution on S.

Table 2. Multiplication table of S with the values of the involution $*$ in the rightmost column

$\cdot$	s_1	s_2	s_3	s_4	$*$
s_1	s_1	s_2	s_3	s_4	s_1
s_2	s_2	s_2	s_2	s_4	s_2
s_3	s_3	s_2	s_1	s_4	s_3
s_4	s_4	s_4	s_4	s_2	s_4

Let $(\mathbb{Z}[S],*)$ be the $*$-semigroup ring of S over the integers where $*$ is the involution induced by S in $\mathbb{Z}[S]$ such that $XX^* = 0$. Now, let $X = \sum_{i=1}^{4} x_i s_i$ be such that

$$XX^* = (x_1s_1 + x_2s_2 + x_3s_3 + x_4s_4)(x_1s_1^* + x_2s_2^* + x_3s_3^* + x_4s_4^*) = 0$$

$$x_1^2s_1 + x_1x_2s_2 + x_1x_3s_3 + x_1x_4s_4$$

$$+x_2x_1s_2 + x_2^2s_2 + x_2x_3s_2 + x_2x_4s_4$$

$$+x_3x_1s_3 + x_3x_2s_2 + x_3^2s_1 + x_3x_4s_4$$

$$+x_4x_1s_4 + x_4x_2s_4 + x_4x_3s_4 + x_4^2s_2 = 0$$

$$s_1(x_1^2 + x_3^2) + s_2(x_1x_2 + x_1x_2 + x_2^2 + x_2x_3 + x_2x_3 + x_4^2)$$

$$+s_3(x_1x_3 + x_1x_3) + s_4(x_1x_4 + x_2x_4 + x_3x_4 + x_1x_4 + x_2x_4 + x_3x_4) = 0$$

Now, put

$$f_1 = x_1^2 + x_3^2 = 0$$
$$f_2 = 2x_1x_2 + x_2^2 + 2x_2x_3 + x_4^2 = 0$$
$$f_3 = 2x_1x_3 = 0$$
$$f_4 = 2x_1x_4 + 2x_2x_4 + 2x_3x_4 = 0$$

It is clear that in $\mathbb{Z}$ if $x_1^2 + x_3^2 = 0$, then $x_1 = x_3 = 0$. By substituting into f_2, we get $x_2^2 + x_4^2 = 0$, thus $x_2 = x_4 = 0$. So, the solution of $XX^* = 0$ in $(\mathbb{Z}[S],*)$ is trivial in this case. By Corollary 3.5, we thus conclude that $(S,*)$ is $*$-embeddable.

6. Computational Analysis

In the examples presented in Sections 4 and 5, the solutions to $XX^* = 0$ in $\mathbb{Z}[S]$ are either trivial or have nonzero coefficients taking only the values ± 1. Our computational analysis using GAP programs finds such a solution in every nontrivial case among the commutative P $*$-semigroups of orders 3, 4, and 5. We developed a two-stage computational procedure, consisting of Algorithms 1 and 2.

Algorithm 1 generates all P $*$-semigroups of a prescribed finite order n. It first enumerates all commutative multiplication tables and retains those that satisfy associativity. For each associative table, every permutation of the underlying set is tested as a candidate P-involution. A permutation is accepted precisely when it satisfies $(a^*)^* = a, (ab)^* = b^*a^*$, together with the properness condition

$$aa^* = ab^* = bb^* \Rightarrow a = b.$$

Each pair consisting of an associative multiplication table and an accepted involution is then recorded as a P $*$-semigroup.

Algorithm 2 takes the output of Algorithm 1 and translates the equation $XX^* = 0$ into a system of homogeneous quadratic equations in the coefficients of X. It then applies the following steps in order:

(1) a sum-of-squares reduction to force variables to be zero;
(2) searches for nonzero solutions modulo 2, 3, 5, and 7;
(3) an exact search for nonzero solutions with coordinates in $\{-1, 0, 1\}$.

If the system has only the zero solution modulo a prime p, then it has only the zero solution over $\mathbb{Z}$. Conversely, the existence of a nonzero solution modulo p does not imply the existence of a nonzero integer solution, so such a modular solution is used only to prevent an immediate triviality certification.

Algorithm 1 Generation of finite P $*$-semigroups

Require: A positive integer n
Ensure: All P $*$-semigroups of order n

Initialize an empty list of P $*$-semigroups.
Generate all symmetric $n \times n$ multiplication tables T.
for each multiplication table T **do**
    if T is associative **then**
        Consider every permutation $*$ of 1, . . . , n as a candidate involution.
        for each permutation $*$ **do**
            if $(a^*)^* = a$ for every a **then**
                if $(ab)^* = b^*a^*$ for every a, b **then**
                    if $aa^* = ab^* = bb^*$ implies $a = b$ for every distinct a, b **then**
                        Record $(T,*)$ as a P $*$-semigroup.
                    end if
                end if
            end if
        end for
    end if
end for
return the list of all recorded P $*$-semigroups

A system is classified as TRIVIAL whenever one of the first two tests proves that only the zero-integer solution is possible, and as NONTRIVIAL when the third step produces an explicit nonzero solution. Systems for which none of these steps give a conclusion are recorded as INCONCLUSIVE.

Algorithm 2 Testing triviality of $XX^* = 0$ in the semigroup ring of P $*$-semigroups

Require: A positive integer n and a P $*$-semigroup of order n
Ensure: A classification as TRIVIAL, NONTRIVIAL, or INCONCLUSIVE.
Construct the original system of equations corresponding to $XX^* = 0$.
Apply the sum-of-squares reduction to the equations.
if all n variables are forced to be zero **then**
    return TRIVIAL with reason "Sum-of-squares".
end if
for $p \in \{2, 3, 5, 7\}$ **do**
    Search all vectors $\mathbf{v} \in \mathbb{Z}_p^n$ satisfying the original equations.
    if the only solution $\mathbf{v} = \mathbf{0}$ **then**
        return TRIVIAL with reason "Mod p".
    end if
end for
Search all vectors $\mathbf{v} \in \{-1, 0, 1\}^n \setminus \{0\}$ satisfying the original equations.
if such a vector **v** is found **then**
    return NONTRIVIAL with reason "Exact search" and solution **v**.
end if
return INCONCLUSIVE with reason "No conclusion from previous steps".

It is important to note that the three steps in Algorithm 2 are sufficient to decide triviality for all P $*$-semigroups of orders 3, 4, and 5. In particular, Algorithm 2 returned no INCONCLUSIVE cases among the 5072 semigroups considered. The results are recorded in Table 3. Applying Corollary 3.5, we obtain that most of these P $*$-semigroups are $*$-embeddable in their corresponding semigroup rings $\mathbb{Z}[S]$.

Table 3. Computational classification of all 5072 commutative P $*$-semigroups according to the outcome of Algorithm 2

Order	P $*$-semigroups	TRIVIAL	NONTRIVIAL	INCONCLUSIVE
3	27	27	0	0
4	320	308	12	0
5	4725	4605	120	0

For all 5072 cases, Algorithm 2 returned either TRIVIAL or NONTRIVIAL; no case was classified as INCONCLUSIVE. In 4940 cases, the algorithm certifies that $XX^* = 0$ has only the trivial solution in $\mathbb{Z}[S]$. Hence, by Corollary 3.5, these semigroups are $*$-embeddable in their corresponding semigroup rings $\mathbb{Z}[S]$. In the remaining 132 cases, the algorithm identifies an explicit nontrivial solution. Consequently, the corresponding semigroup ring $\mathbb{Z}[S]$ is not a P $*$-ring. This observation alone does not determine whether these semigroups are P $*$-embeddable in some other P $*$-ring.

For every case classified as NONTRIVIAL, the exact search produces a nonzero solution whose coefficients lie in $\{-1, 0, 1\}$ and sum to 0. Thus, in all NONTRIVIAL cases considered here, a solution with this particularly simple coefficient pattern is available.

7. Conclusion

Our computational results for commutative $P*$-semigroups of orders 3, 4, and 5 completely determine the triviality of $XX^* = 0$ in the corresponding semigroup rings. Of the 5072 $P*$-semigroups considered, 4940 are $*$-embeddable in $\mathbb{Z}[S]$ by Corollary 3.5, while the remaining 132 have corresponding semigroup rings that are not $P*$-rings.

In every nontrivial case, the exact search yields a solution with coefficients in $\{-1,0,1\}$ whose sum is 0. This suggests the question of whether every nontrivial solution of $XX^* = 0$ admits such a representative. We leave this as a conjectural direction for future work.

It is also natural to investigate proper $*$-semigroups of order 6 and higher, both to test whether the observed pattern persists and to further understand the relationship between proper $P*$-semigroups and $*$-embeddability.

References

[1] A.A. Shehadah, "Proper embeddability of inverse semigroups," Bull. Austral. Math. Soc. Vol. 34, 383-387, 1986.

[2] E.W.H. Lee, "Embedding finite involution semigroups in matrices with transposition," Discrete Applied Mathematics, Elsevier. Vol. 340, 327-330, 2023.

[3] A.A. Shehadah, "A counterexample on $*$-embeddability into proper $*$-rings," Semigroup Forum, Springer. Vol. 34, 121-123, 1986.

[4] A.A. Shehadah, "Formal complexity of inverse semigroups ring," Bull. Austral. Math. Soc. Vol. 41, 343-346, 1990.

[5] A.A. Shehadah, "n-proper maximal involutions on semigroups and rings," Japanese Association of Mathematical Sciences. Vol. 46, 293-296, 1997.

[6] A.A. Shehadah, "Embedding theorems for semigroups with involution," Ph.D. Thesis. Purdue University, West Lafayette, Ind., U.S.A. 1982.

[7] A.A. Abdelkarim, "Regular proper $*$-embedding of proper $*$-semigroups and rings," J. Semigroup Theory Appl., ID: 2013:2, 2013.

[8] The GAP Group, "GAP–groups, algorithms, and programming," Version 4.16.1, 2026.

[9] M. Petrich, "Introduction to semigroups," Merrill Publishing Company (a division of A Bell & Howell Company), Columbus Ohio, 1973.

[10] M.P. Drazin, "Natural structures on semigroups with involution," Bull. Amer. Math. Soc. Vol. 84, 139-141, 1978.

[11] D. Easdown, W.D. Munn, "On semigroups with involution," Bull. Amer. Math. Soc., Cambridge University Press. Vol. 48, 93-100, 1993.