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On Neighborhoods of a New Class of Meromorphic Multivalent Functions with Negative Coefficients

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Abstract—In this paper, we introduce an extension of the generalized derivative operator $I_v^m(\lambda_1,\lambda_2,l,n)f(z)$, from the field of multivalent analytic functions in the open unit disc to the field of meromorphic multivalent functions in the punctured open unit disc. Then we introduce and systematically study a new class $\mathcal{K}_v^m(\lambda_1,\lambda_2,l,n,\varpi,\gamma)$ of meromorphic multivalent functions in the punctured open unit disc by means of a new extended generalized derivative operator $I_v^m(\lambda_1,\lambda_2,l,n)f(z)$. This class of functions is defined by a condition involving the proposed extended operator, which gave this class new geometric and analytical properties. Membership in this class is characterized by coefficient estimates. In addition, the inclusion and neighborhood properties of this class of functions are investigated.

Keywords—Coefficient estimates; Generalized derivative operator; Meromorphic multivalent functions; Neighborhood properties

1. Introduction

The Riemann mapping theorem in complex analysis introduced a fundamental idea for studying complex functions, leading to the development of a new branch known as geometric function theory (GFT), in which the geometric properties of analytic functions are investigated. Later, in 1907, Paul Koebe initiated systematic work on univalent functions [1]. Subsequently, various geometric subclasses such as starlike, convex, close-to-convex, spiral-like, and multivalent functions [2] were introduced and extensively studied by researchers including M. S. Robertson, Karl Löwner, Wilfred Kaplan, Ladislav Špaček, and Robert J. Libera.

The classical study of the subject of analytic univalent functions has been engaging the attention of researchers at least till as early as 1907. This has been growing vigorously with added research. This field captioned as geometric function theory is found to be a mixing or an interplay of geometry and analysis. Despite the classical nature of the subject, unlike contemporary areas, this field has been fascinating researchers, with stress on the interest based on investigations by function theorists. The main ingredient motivating this line of thought is based on the famous conjecture called the Bieberbach conjecture or coefficient problem offering vast scope for development from 1916, till a positive settlement in 1985 by de Branges where innumerable results were obtained based on this problem. Multivalent functions are a natural generalization of univalent functions. There are several results that generalize classical results on univalent functions, the study of multivalent functions expands the scope of GFT, allowing mathematicians to explore families of functions with richer structural behaviors and more complex mapping properties. One of the research methods that has accompanied multivalent functions since their appearance is the study of subclasses of functions defined using certain operators such as derivative operators [3–8], and obtain neighborhood properties [9-14].

The theory of meromorphic multivalent functions extends the classical theory of univalent analytic functions by allowing functions with multiple valency and poles in the domain, thereby providing a richer framework for analysis. Within this context, neighborhood problems have been extensively studied to investigate how small variations in coefficients affect membership in specific subclasses. Such studies are closely related to convolution (Hadamard product), derivative operators, which are frequently used to define and investigate new function classes [15-17]. As a contribution to the development of the theory of multivalent meromorphic functions and the investigation of their structural properties, many researchers have focused on extending generalized derivative operators, so as to be applied to the class of multivalent meromorphic functions in the punctured open unit disc [18-22]. Moreover, these extended generalized derivative operators have been employed to define new subclasses of multivalent meromorphic functions and to investigate their inclusion relationships and neighborhood properties [23-27].

Despite these efforts, there remains a need for further research to gain a deeper understanding of the analytic and geometric behavior of multivalent meromorphic functions associated with newly developed generalized derivative operators. In particular, there is a continuing interest in constructing further extensions of generalized derivative operators that are applicable to the class of multivalent meromorphic functions. According to that, the present study aims to contribute to this theoretical research field, by extending a generalized derivative operator [28] to the class of multivalent meromorphic functions, and introducing a new subclass of multivalent meromorphic functions in the punctured open unit disc defined in terms of the proposed operator.

2. Preliminaries

We first denote by $\mathcal{K}$ the class of analytic functions in the open unit disc

$$E = \{z : z \in \mathbb{C}, |z| < 1\},$$

which have the following form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad z \in E.$$

Let $\mathcal{A}$ denote the subclass of $\mathcal{K}$ consisting of functions $f(z)$ in E and satisfy the normalization condition:

$$f(0) = f'(0) - 1 = 0.$$

For two functions $f(z)$ given by (1) and $g(z)$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \qquad z \in E,$$

the convolution (or, equivalently, the Hadamard product) is defined below

$$.(f * g)(z) = z + \sum_{k=2}^{\infty} a^k b_k z^k$$

Let $(x)_k$denote the pochhamer symbol defined by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} 1 & for\ k = 0, x \in \mathbb{C}^* = \mathbb{C}/\{0\} \\ x(x+1)(x+2)\ldots(x+k-1) & for\ k \in \mathbb{N}, x \in \mathbb{C}. \end{cases}$$

Using the Hadamarad product, Amer and Darus [29] have recently introduced a new generalized derivative operator $I^m(\lambda_1, \lambda_2, l, n)f(z)$ as follows:

For $m, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\lambda_2 \geq \lambda_1 \geq 0, l \geq 0$. Let

$$\phi^m(\lambda_1, \lambda_2, l)(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} z^k.$$

We define the generalized derivative operator $I^m(\lambda_1, \lambda_2, l, n)f(z): \mathcal{A} \to \mathcal{A}$ by:

$$I^m(\lambda_1, \lambda_2, l, n)f(z) = \phi^m(\lambda_1, \lambda_2, l)(z) * R^n f(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}\left(1+\lambda_2(k-1)\right)^m} c(n,k)\, a_k z^k,$$

where $R^n f(z)$ is the Rushewehyh derivative operator which given by

$$R^n f(z) = z + \sum_{k=2}^{\infty} c(n,k)\, a_k z^k, \text{ and } c(n,k) = \frac{(n+1)_{k-1}}{(1)_{k-1}},$$

Let $\mathcal{A}(v)$denote the class of functions of the form:

$$f(z) = z^v + \sum_{k=v+1}^{\infty} a_k z^k, \quad (v \in \mathbb{N}).$$

which are analytic multivalent in the open unit disk E.

The authors [28], extended the generalized derivative operator $I^m(\lambda_1, \lambda_2, l, n)f(z)$ for a class $\mathcal{A}(v)$ as follows:

For a function $f(z) \in \mathcal{A}(v)$, we define the extended generalized derivative operator $I_v^m(\lambda_1, \lambda_2, l, n)f(z)$ by

$$I_v^m(\lambda_1, \lambda_2, l, n)f(z) = z^v + \sum_{k=v+1}^{\infty} \frac{(v+\lambda_1(k-v)+l)^{m-1}}{(v+l)^{m-1}\left(1+\lambda_2(k-v)\right)^m} c^v(n,k)\, a_k, \tag{1}$$

where,

$$c^v(n,k) = \binom{n+k-1}{k-v} = \frac{\Gamma(k+n)}{(k-v)!\,\Gamma(v+n)}, \text{ and } m, n \in \mathbb{N}_0, v \in \mathbb{N}.$$

Special cases of this operator include:

- $I_1^m(\lambda_1, \lambda_2, l, n)f(z) = I^m(\lambda_1, \lambda_2, l, n)f(z)$, defined by Amer and Darus [29].
- $I_1^{m+1}(0,0,0,n)f(z) = R^n f(z)$, the Rusheweyh derivative operator [24].
- $I_1^{m+1}(1,0,0,0)f(z) = S^n f(z)$, the Salagean derivative operator [27].
- $I_1^2(\lambda_1, 0,0,n)f(z) = R_\lambda^n f(z)$, the generalized Rusheweyh derivative operator [30].
- $I^{m+1}(\lambda_1, 0,0,0)f(z) = S_\beta^n f(z)$, the generalized Salagean derivative operator introduced by Al-Oboudi [3].
- $I^m(\lambda_1, 0, l, n)f(z) = I^m(\lambda, \beta, l)f(z)$ the Catas derivative operator [12].
- $I_1^m(\lambda_1, \lambda_2, 0, n)f(z) = \mu_{\lambda_1,\lambda_2}^{n,m} f(z)$, defined by AL-Abbad and Darus [31].
- $I_v^{m+1}(1,0,0,0)f(z) = S_p^n f(z)$, defined by Eker and Seker [15].
- $I_v^{m+1}(0,0,0,n)f(z) = R^{\lambda,p} f(z)$, introduced Raina and Srivastava [26].
- $I_v^{m+1}(0,0,0,m+p-1)f(z) = D^{n+p-1} f(z)$, defined by Kumar and Shukla [20].
- $I_v^{m+1}(\lambda_1, 0,0,0)f(z) = D_{\delta,p}^n f(z)$, introduced by Bulut [11].

- $I_v^m(\lambda_1, 0, l, 0)f(z) = I_p^n(\lambda, l)f(z)$, the Catas extended multiplier transformation [13].

Additionally, we let $\mathcal{F}(v)$ denote the class of all meromorphic functions of the form

$$f(z) = \frac{1}{z^v} + \sum_{k=v}^{\infty} a_k z^k,$$

which are analytic and multivalent in the punctured open unit disk in complex plane

$$E^* = \{z : z \in \mathbb{C}, 0 < |z| < 1\}.$$

Also, $\mathcal{M}(v)$ denote the subclass of $\mathcal{F}(v)$ that contains all functions of the form

$$f(z) = \frac{1}{z^v} - \sum_{k=v}^{\infty} a_k z^k, \qquad (a_k \geq 0;\ v, k \in \mathbb{N}, k \geq v,\ 0 < |z| < 1). \tag{2}$$

That is, $\mathcal{M}(v)$ the subclass of all analytic and meromorphic multivalent functions (2) in E^* with negative coefficients. If $f(z) \in \mathcal{M}(v)$, then (1) becomes

$$I_v^m(\lambda_1, \lambda_2, l, n)f(z) = \frac{1}{z^v} - \sum_{k=v}^{\infty} \frac{(v+\lambda_1(k+v)+l)^{m-1}}{(v+l)^{m-1}\big(1+\lambda_2(k+v)\big)^m}\, c_v^{\varkappa}(n,k)\, a_k z^k \tag{3}$$

where, $c_v^{\varkappa}(n,k) = \binom{n+v+k-1}{k}$.

For simplification, we write Equation (3) in the form

$$I_v^m(\lambda_1, \lambda_2, l, n)f(z) = \frac{1}{z^v} - \sum_{k=v}^{\infty} \phi_k^m(\lambda_1, \lambda_2, l, n)(z)\ \ a_k z^k\ ,$$

where,

$$\phi_k^m(\lambda_1, \lambda_2, l, n)(z) = \frac{c_v^{\varkappa}(n,k)(v+\lambda_1(k+v)+l)^{m-1}}{(v+l)^{m-1}\big(1+\lambda_2(k+v)\big)^m}\,. \tag{4}$$

Definition 2.1

Let $v \in \mathbb{N}$, $0 \leq \varpi < 1$, and $\gamma > 0$. The class $\mathcal{K}_v^m(\lambda_1, \lambda_2, l, n, \varpi, \gamma)$ consists of all functions $f \in \mathcal{M}(v)$, of the form (2), such that

$$\sum_{k=v}^{\infty} \phi_v^m(\lambda_1, \lambda_2, l, n)(z)[k - 1 + \varpi(v+1)]k\ \ a_k < v(v+1)(1-\varpi).$$

Where, $\phi_v^m(\lambda_1, \lambda_2, l, n)(z)$ given by (4),

and

$$|(U_v^m f)(z)| < \gamma |(V_v^m f)(z)|,$$

for $0 < |z| < 1$,

$$(U_v^m f)(z) = \frac{z^v (I_v^m(\lambda_1,\lambda_2,l,n)f)''(z)}{(I_v^m(\lambda_1,\lambda_2,l,n)f)'(z)} + (v+1)z^{v-1},$$

and

$$(V_v^m f)(z) = \frac{z^v (I_v^m(\lambda_1,\lambda_2,l,n)f)''(z)}{(I_v^m(\lambda_1,\lambda_2,l,n)f)'(z)} + \varpi(v+1)z^{v-1}.$$

Remark 2.1

Setting $\lambda_1 = \lambda_2 = l = n = m = 0, v = 1, \varpi = \alpha$ and $\gamma = 1$. The class $\mathcal{K}_v^m(\lambda_1, \lambda_2, l, n, \varpi, \gamma)$ agrees with $\Lambda_{1,\kappa}^*(\alpha, -1,1,0)$ in [17].

Remark 2.2

Setting $\lambda_1 = \lambda_2 = l = 0, m = n, \upsilon = 1, \varpi = 0$ and $\gamma = 1$. The class $\mathcal{K}_\upsilon^m(\lambda_1, \lambda_2, l, n, \varpi, \gamma)$ agrees with $\mathrm{T}_1^*(0,1,-1,1,n)$in [5].

In the following section, we present the coefficient bounds for the class $\mathcal{K}_\upsilon^m(\lambda_1, \lambda_2, l, n, \varpi, \gamma)$**.**

3. Coefficient Estimates

The following theorem provides a necessary and sufficient condition for a function to belong to the class $\mathcal{K}_\upsilon^m(\lambda_1, \lambda_2, l, n, \varpi, \gamma)$ in terms of its coefficients.

Theorem 3.1

A function $f \in \mathcal{M}(\upsilon)$, given by (2), is in a class $\mathcal{K}_\upsilon^m(\lambda_1, \lambda_2, l, n, \varpi, \gamma)$ if and only if

$$\sum_{k=\upsilon}^{\infty} \phi_k^m(\lambda_1, \lambda_2, l, n)(z)[\upsilon + k + \gamma(k-1) + \varpi\,\gamma(\upsilon+1)]k\ a_k < \gamma(1-\varpi)\upsilon(\upsilon+1), \qquad (5)$$

The estimate is sharp for any of the functions

$$f_k(z) = \frac{\gamma(1-\varpi)\upsilon(\upsilon+1)}{\phi_k^m(\lambda_1,\lambda_2,l,n)(z)[\upsilon+\gamma(k-1)+\varpi\,\gamma(\upsilon+1)]k},$$

where $\phi_\upsilon^m(\lambda_1, \lambda_2, l, n)(z)$ given by (4).

Proof

Suppose that $f \in \mathcal{M}(\upsilon)$, be given by (2), and fix z. To establish that the function $f(z)$ belongs to the class $\mathcal{K}_\upsilon^m(\lambda_1, \lambda_2, l, n, \varpi, \gamma)$, we verify both conditions stated in Definition 2.1, assume $I_\upsilon^m(\lambda_1, \lambda_2, l, n) = I_\upsilon^m$, $\phi_k^m(\lambda_1, \lambda_2, l, n)(z) = \phi_k^m$.

We first note that

$$|(U_\upsilon^m f)(z)(I_\upsilon^m f)'(z)| = |z^\upsilon (I_\upsilon^m f)''(z) + (I_\upsilon^m f)'(z)(\upsilon+1)z^{\upsilon-1}|,$$
$$= |-\sum_{k=\upsilon}^{\infty} \phi_k^m\,(\upsilon+k)ka_k\,z^{\upsilon+k-2}|. \qquad (6)$$

Since $|z| < 1$, by the triangle inequality for complex-valued functions, we have from (6)

$$|(U_\upsilon^m f)(z)(I_\upsilon^m f)'(z)| < \sum_{k=\upsilon}^{\infty} \phi_k^m\,(\upsilon+k)ka_k, \qquad (7)$$

by using the same strategy, we have

$$|(V_\upsilon{}^m f)(z)(I_\upsilon^m f)'(z)| = |z^\upsilon (I_\upsilon^m f)''(z) + \varpi(\upsilon+1)z^{\upsilon-1}(I_\upsilon^m f)'(z)|,$$
$$= |(1-\varpi)\upsilon(\upsilon+1)z^{-2} - \sum_{k=\upsilon}^{\infty} \phi_k^m\,[k-1+\varpi(\upsilon+1)]ka_k\,z^{\upsilon+k-2}|$$
$$> (1-\varpi)\upsilon(\upsilon+1) - \sum_{k=\upsilon}^{\infty} \phi_k^m\,(\upsilon+k)[k-1+\varpi(\upsilon+1)]ka_k. \qquad (8)$$

Now, if (5) holds, and by relating this to equations (7) and (8)

$$0 \le \sum_{k=\upsilon}^{\infty} \phi_k^m\,(\upsilon+k)ka_k$$
$$\le \gamma(1-\varpi)\upsilon(\upsilon+1) - \gamma\sum_{k=\upsilon}^{\infty} \phi_k^m\,(\upsilon+k)[k-1+\varpi(\upsilon+1)]ka_k,$$

which implies, the first condition of definition 2.1, moreover

$$|(U_\upsilon^m f)(z)(I_\upsilon^m f)'(z)| - \gamma|(V_\upsilon{}^m f)(z)(I_\upsilon^m f)'(z)|$$
$$< \sum_{k=\upsilon}^{\infty} \phi_k^m\,(\upsilon+k)[\upsilon+k+\gamma(k-1)+\varpi\,\gamma(\upsilon+1)]ka_k - \gamma(1-\varpi)\upsilon(\upsilon+1) \le 0.$$

Thus, $f \in \mathcal{K}_\upsilon^m(\lambda_1, \lambda_2, l, n, \varpi, \gamma)$.

Conversely, assuming $f(z) \in \mathcal{K}_{\upsilon}^{m}(\lambda_1,\lambda_2,l,n,\varpi,\gamma)$, we proceed to establish that inequality (5) is satisfied, we first note that

$$\left|\frac{(U_{\upsilon}^{m}f)(z)}{(V_{\upsilon}^{m}f)(z)}\right| = \left|\frac{\sum_{k=\upsilon}^{\infty}\phi_k^m(\upsilon+k)ka_kz^{\upsilon+k}}{(1-\varpi)\upsilon(\upsilon+1)-\sum_{k=\upsilon}^{\infty}\phi_k^m[k-1+\varpi(\upsilon+1)]ka_kz^{\upsilon+k}}\right| < \gamma,$$

From the above relation, it follows that

$$Re\left[\frac{\sum_{k=\upsilon}^{\infty}\phi_{\upsilon}^m(\upsilon+k)ka_kz^{\upsilon+k}}{(1-\varpi)\upsilon(\upsilon+1)-\sum_{k=\upsilon}^{\infty}\phi_k^m[k-1+\varpi(\upsilon+1)]ka_kz^{\upsilon+k}}\right] < \gamma.$$

Choosing $z = r \in \mathbb{R}$ and letting $r \to 1^-$ yields

$$\frac{\sum_{k=\upsilon}^{\infty}\phi_k^m(\upsilon+k)ka_k}{(1-\varpi)\upsilon(\upsilon+1)-\sum_{k=\upsilon}^{\infty}\phi_k^m[k-1+\varpi(\upsilon+1)]ka_k} < \gamma.$$

As the denominator on the left-hand side of this inequality is positive, (5) follows.
Finally. It is straightforward to verify that the estimate (5) is sharp for the functions $f_k(z)$, $(k \in \mathbb{N}, k \geq \upsilon)$.

Corollary 3.1

If $f \in \mathcal{K}_{\upsilon}^{m}(\lambda_1,\lambda_2,l,n,\varpi,\gamma)$, given by (2), then

$$\sum_{k=\upsilon}^{\infty} a_k \leq \frac{\gamma(1-\varpi)(\upsilon+1)}{\phi_{\upsilon}^m(\lambda_1,\lambda_2,l,n)[2\upsilon+\gamma(\upsilon-1)+\varpi\,\gamma(\upsilon+1)]}.$$

This estimate is sharp for function $f_{\upsilon}(z)$ in theorem 3.1.

4. Neighborhood properties

Definition 4.1

Suppose $f \in \mathcal{M}(\upsilon)$ is given by (2), and let $0 \leq \delta < 1$. The $N_{\upsilon,\delta}$_$neighborhood$ of f is defined by

$$N_{\upsilon,\delta}(f) = \left\{g(z) = \frac{1}{z^{\upsilon}} - \sum_{k=\upsilon}^{\infty} b_k z^k \in \mathcal{M}(\upsilon): \sum_{k=\upsilon}^{\infty} k|a_k - b_k| \leq \delta\right\}.$$

If $h(z) = z^{-\upsilon}$, then

$$N_{\upsilon,\delta}(h) = \left\{g(z) = \frac{1}{z^{\upsilon}} - \sum_{k=\upsilon}^{\infty} b_k z^k \in \mathcal{M}(\upsilon): \sum_{k=\upsilon}^{\infty} k|b_k| \leq \delta\right\}.$$

Similarly. The $\widetilde{N}_{\upsilon,\delta} - neighborhood$ of f is defined by

$$\widetilde{N}_{\upsilon,\delta}(f) = \left\{g(z) = \frac{1}{z^{\upsilon}} - \sum_{k=\upsilon}^{\infty} b_k z^k \in \mathcal{M}(\upsilon): \sum_{k=\upsilon}^{\infty} \mu_k|a_k - b_k| \leq \delta\right\},$$

where

$$\mu_k = \frac{\phi_{\upsilon}^m(\lambda_1,\lambda_2,l,n)[\upsilon+k+\beta(k-1)+\alpha\beta(\upsilon+1)]k}{\gamma(1-\varpi)\upsilon(\upsilon+1)} \qquad ,(k \in \mathbb{N}, k \geq \upsilon).$$

Definition 4.2

Let $0 \leq \sigma < 1$ the function $f \in \mathcal{M}(\upsilon)$ is said to lie in the class $\mathcal{K}_{\upsilon,\sigma}^{m}(\lambda_1,\lambda_2,l,n,\varpi,\gamma)$, whenever there exists $g(z) \in \mathcal{K}_{\upsilon}^{m}(\lambda_1,\lambda_2,l,n,\varpi,\gamma)$ such that

$$\left|\frac{g(z)}{f(z)} - 1\right| < 1 - \sigma; \qquad (0 < |z| < 1).$$

Theorem 4.1

Let $f \in \mathcal{K}_{\upsilon}^{m}(\lambda_1, \lambda_2, l, n, \varpi, \gamma)$ be given by (4). If $\mu_\upsilon > 1$, $0 \leq \delta \leq \frac{\mu_\upsilon - 1}{\mu_\upsilon} < 1$ and

$$\sigma = \left[1 - \frac{\mu_\upsilon \delta}{\mu_\upsilon - 1}\right] \geq 0,$$

then $N_{\upsilon,\delta}(f) \subset \mathcal{K}_{\upsilon,\sigma}^{m}(\lambda_1, \lambda_2, l, n, \varpi, \gamma)$.

Proof

Pick $g(z) \in \mathcal{M}(\upsilon)$ given by

$$g(z) = \frac{1}{z^\upsilon} - \sum_{k=\upsilon}^{\infty} b_k z^k \quad , (b_k \geq 0;\ k \in \mathbb{N}, k \geq \upsilon\ ). \tag{9}$$

Assume also $\sum_{k=\upsilon}^{\infty} k|a_k - b_k| \leq \delta$, then

$$\sum_{k=\upsilon}^{\infty} |a_k - b_k| \leq \delta \tag{10}$$

Now, from corollary 3.1, we have

$$\sum_{k=\upsilon}^{\infty} a_k \leq \frac{\gamma(1-\varpi)(\upsilon+1)}{\phi_\upsilon^m(\lambda_1,\lambda_2,l,n)[2\upsilon+\gamma(\upsilon-1)+\varpi\,\gamma(\upsilon+1)]} = {\mu_\upsilon}^{-1} \quad , \tag{11}$$

and, by condition $\mu_\upsilon > 1$, we get ${\mu_\upsilon}^{-1} < 1$.

Thus, the Equation (11) becomes

$$\sum_{k=\upsilon}^{\infty} a_k \leq {\mu_\upsilon}^{-1} < 1. \tag{12}$$

Hence

$$\left|\frac{g(z)}{f(z)} - 1\right| < \frac{\sum_{k=\upsilon}^{\infty}|a_k - b_k||z|^{\upsilon+k}}{1 - \sum_{k=\upsilon}^{\infty} a_k |z|^{\upsilon+k}},$$

$$\leq \frac{\sum_{k=\upsilon}^{\infty}|a_k - b_k|}{1 - \sum_{k=\upsilon}^{\infty} a_k} \ . \tag{13}$$

Using equations (10), (12) in (13), we get

$$\left|\frac{g(z)}{f(z)} - 1\right| < \frac{\mu_\upsilon \delta}{\mu_\upsilon - 1} = 1 - \sigma, \qquad 0 < |z| < 1.$$

Theorem 4.2

Suppose $f \in \mathcal{K}_{\upsilon}^{m}(\lambda_1, \lambda_2, l, n, \varpi, \gamma)$ be given by (2). If

$$\delta = 1 - \sum_{k=\upsilon}^{\infty} \mu_k a_k,$$

then $\widetilde{N}_{\upsilon,\delta}(f) \subset \mathcal{K}_{\upsilon}^{m}(\lambda_1, \lambda_2, l, n, \varpi, \gamma)$, this result is sharp, in the sense that δ cannot be increased.

Proof

It follows immediately from Theorem 3.1 that

$$\sum_{k=\upsilon}^{\infty} \mu_k a_k = \zeta \leq 1,$$

Let $g \in \widetilde{N}_{\upsilon,\delta}(f)$, be given by (9). We may write

$$\sum_{k=\upsilon}^{\infty} \mu_k b_k \leq \sum_{k=\upsilon}^{\infty} \mu_k a_k + \sum_{k=\upsilon}^{\infty} \mu_k |b_k - a_k| \leq \zeta + \delta = 1.$$

Thus, by theorem 3.1 $g \in \mathcal{K}_{\upsilon}^{m}(\lambda_1, \lambda_2, l, n, \varpi, \gamma)$.

To prove the sharpness of the result, consider the extremal function

$f = f_\upsilon \in \mathcal{K}_{\upsilon}^{m}(\lambda_1, \lambda_2, l, n, \varpi, \gamma)$ from Theorem 3.1

$$f(z)=\frac{1}{z^{\upsilon}}-\frac{1}{\mu_{\upsilon}}z^{\upsilon}, \qquad (z\in E^{*})$$

note that, for this f, we have $\delta=0$.
Choose $0<\delta^{*}<1$, and define $g\in\mathcal{M}(\upsilon)$ by

$$g(z)=\frac{1}{z^{\upsilon}}-\frac{1+\delta^{*}}{\mu_{\upsilon}}z^{\upsilon}.$$

Thus $g\in\widetilde{N}_{\upsilon,\delta^{*}}(f)$:

$$\mu_{\upsilon}\left|\frac{1}{\mu_{\upsilon}}-\frac{1+\delta^{*}}{\mu_{\upsilon}}\right|=\delta^{*}.$$

However,

$$\frac{\phi_{\upsilon}^{m}(\lambda_1,\lambda_2,l,n)[2\upsilon+\gamma(k-1)+\varpi\,\gamma(\upsilon+1)]}{\gamma(1-\varpi)(\upsilon+1)}\frac{1+\delta^{*}}{\mu_{\upsilon}}=1+\delta^{*}>1,$$

and theorem 3.1, reveals that $g\notin\mathcal{K}_{\upsilon}^{m}(\lambda_1,\lambda_2,l,n,\varpi,\gamma)$.

5. Inclusion Relations Involving $N_{\upsilon,\delta}(h)$

Theorem 5.1

If

$$\delta=\frac{\gamma\upsilon^{2}(1-\varpi)(\upsilon+1)}{\phi_{\upsilon}^{m}(\lambda_1,\lambda_2,l,n)[2\upsilon+\gamma(\upsilon-1)+\upsilon\,\varpi\,\gamma(\upsilon+1)]\upsilon},$$

then $\mathcal{K}_{\upsilon}^{m}(\lambda_1,\lambda_2,l,n,\varpi,\gamma)\subset N_{\upsilon,\delta}(h)$.

Proof

Let $f(z)\in\mathcal{K}_{\upsilon}^{m}(\lambda_1,\lambda_2,l,n,\varpi,\gamma)$, then, Theorem 3.1 immediately yields

$$\phi_{\upsilon}^{m}(\lambda_1,\lambda_2,l,n)(z)[2\upsilon+\gamma(\upsilon-1)+\varpi\,\gamma(\upsilon+1)]\sum_{k=\upsilon}^{\infty}k\ a_k<\gamma(1-\varpi)\upsilon(\upsilon+1)\ ,$$

so that

$$\sum_{k=\upsilon}^{\infty}k\ a_k\leq\frac{\gamma\upsilon^{2}(1-\varpi)(\upsilon+1)}{\phi_{\upsilon}^{m}(\lambda_1,\lambda_2,l,n)[2\upsilon+\gamma(\upsilon-1)+\upsilon\,\varpi\,\gamma(\upsilon+1)]\upsilon}=\delta,$$

Thus, by Definition 2.1, we get $f(z)\in N_{\upsilon,\delta}(h)$.

6. Conclusions

This paper presents a systematic investigation of a new class of meromorphic multivalent functions $\mathcal{K}_{\upsilon}^{m}(\lambda_1,\lambda_2,l,n,\varpi,\gamma)$ defined in the punctured open unit disc by means of the extended generalized derivative operator. Several fundamental results associated with this class were derived, including coefficient estimates that provide sufficient characterization for class membership. Furthermore, the inclusion relations and neighborhood properties related to this class were studied, contributing to a deeper understanding of the analytic structure and geometric properties of these functions. The results obtained in this study suggest several directions for future research:

- New subclasses: The proposed extended operator, defined on the class of meromorphic multivalent functions in the punctured unit disk, can be employed to introduce new subclasses and investigate their inclusion relations and neighborhood properties.

- Geometric properties: Further geometric properties of the studied class can be investigated, including radii of starlikeness and convexity, distortion theorems, growth theorems, and closure properties.
- Extensions to other operators: The methodology developed in this study can be applied to extend other derivative operators from the class of multivalent analytic functions in the open unit disk to the class of meromorphic multivalent functions in the punctured unit disk, followed by an investigation of their associated analytic and geometric properties.

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