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Some Sufficient Conditions Involving Coefficient Inequalities in New Classes

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Abstract—In the present study, we introduce new classes of univalent functions denoted by $V_l^{m,\lambda}(a,b,c,\alpha)$ and $V_l^{m,\lambda}(a,b,c,\alpha,\theta)$ by a linear operator. These classes are related to the classes of starlike and convex functions. This research also discusses several interesting sufficient conditions involving coefficient inequalities for these classes. Additionally, several new results are shown after specializing the parameters employed in main results.

Keywords—Hadamard product; Univalent functions; Starlike functions; Convex functions; Linear operator

الملخص— في البحث الحالي، نقدم فئات جديدة من الدوال أحادية القيمة التي يتم تمثيلها باستخدام مؤثر خطي. هذه الفئات ترتبط بفئات الدوال النجميّة والمحدبة. كما يتناول البحث عدة شروط كافية مثيرة للاهتمام تتعلق بالمتباينات الخاصة بالمعاملات لهذه الفئات. بالإضافة إلى ذلك، يتم عرض نتائج جديدة بعد تخصيص المعاملات المستخدمة في النتائج الأساسية.

الكلمات المفتاحية—حاصل ضرب هادامارد؛ الدوال أحادية القيمة؛ الدوال النجميّة؛ الدوال المحدبة؛ المؤثر الخطي.

1. Introduction

Univalent functions are defined on the open unit disc, and their study, which began in the early 20th century, is an important part of complex analysis. It has noticed that, there has been plenty of work in this field. This function is characterized by the fact that such a function is able to be mapped one-to-one and onto its image. Many scholars investigated some new classes of univalent function such as the classes of convex and starlike functions. For example, Goodman [1] has studied the geometric properties of these classes in detail. Additionally, Kaplan [2] introduced a class of close-to-convex univalent functions and examined their properties. This work opened up a new perspective for the study of operators and classes in this major. Then new generalized operators and classes of analytic and univalent functions have been defined by several studies [3–5]. Many research papers have utilized the same techniques to investigate various problems related to this area, which can be found in the literature [6–15]. Therefore, the authors of this paper are interested in sufficient conditions involving coefficient inequalities for functions in new classes.

Let $\mathcal{A}$ denote the class of functions f in the open unit disc

$$\mathbb{U} = \{z: |z| < 1; z \in \mathbb{C}\},$$

given by the normalized power series

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k ; \quad z \in \mathbb{U}, \tag{1}$$

where a_k is a complex number.

The class of univalent function in $\mathcal{A}$ represented by $\mathcal{S}$ is normalized with the following conditions

$$f(0) = f'(0) - 1 = 0.$$

2. Background

Definition 1 [2]: The Hadamard product (or convolution) of two analytic functions f and g, where f is given by (1) and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, $(z \in \mathbb{U})$ denoted by $f * g$ is defined

$$(f * g)(z) = f(z) * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

In the following, the linear operator $D_l^{m\lambda,}(a,b)f(z)$ was introduced by next definition, also this operator has been used in many of research papers, see [16,17].

Definition 2 [17]: For $f \in \mathcal{A}$ *the* operator $D_l^{m\lambda,}(a,b)f(z)$ *is* defined by $D_l^{m\lambda,}(a,b)f(z): \mathcal{A} \to \mathcal{A}$, and

let $\quad f(z) = \frac{1+l-\lambda}{1+l}\left(\frac{z}{1-z}\right) + \frac{\lambda}{1+l}\left(\frac{z}{(1-z)^2}\right)$, and

$$D_l^{m,\lambda}(a,b)f(z) = \underbrace{\phi(z) * \dots * \phi(z)}_{(m)-times} * zF(a,1;b;z) * f(z),$$

if $(m = 0,1,2,\dots)$, and

$$D_l^{m,\lambda}(a,b)f(z) = \underbrace{\phi(z) * \dots * \phi(z)}_{(-m)-times} * zF(a,1;b;z) * f(z),$$

if $(m = -1,-2,\dots)$, Thus we have

$$D_l^{m,\lambda}(a,b)f(z) := z + \sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m \frac{(a)_{k-1}}{(b)_{k-1}} a_k z^k,$$

where $f \in \mathcal{A}$ and $(z \in \mathbb{U}, b \neq 0,-1,-2,-3,\dots), l, \boldsymbol{\lambda} \geq \mathbf{0}, m \in \mathbb{Z}$ *and*

$(x)_k$ denotes the Pochhammer symbol (or the shifted factorial) defined by:

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} 1 & \text{for } k = 0, x \in \mathbb{C}^* = \mathbb{C}/\{0\} \\ x(x+1)(x+2)\dots(x+k-1) & \text{for } \quad k \in \mathbb{N}, x \in \mathbb{C}, \end{cases}$$

where $\Gamma(x)$ is the gamma function is defined for all complex numbers except the non-positive integers.

This linear operator is the generalized form of the following operators:

- $D_0^{m,0}(a,b)f(z) = D_1^{0,\lambda}(a,b)f(z) = L(a,b)$, see [18].
- $D_0^{0,0}(\beta+1,1)f(z) = D^{\beta}f(z), \beta \geq -1$, see [5].
- $D_0^{m,1}(1,1)f(z) = D^m f(z), m \in \mathbb{N}$, see [3].
- $D_0^{m,1}(1,1)f(z) = D^{m,\lambda}f(z), m \in \mathbb{N}$, see [10].
- $D_l^{m,1}(1,1)f(z) = D_l^{m,\lambda}(a,b)f(z), m \in \mathbb{N}$, see [4].
- $D_0^{0,0}(2,2-\gamma)f(z) = \Omega^{\gamma}f(z) = \Gamma(2-\gamma)z^{\gamma}D_z^{\gamma}f(z)$, see [19].

Definition 3 [1]: A function $f \in \mathcal{A}$ is said to be starlike of order α, $0 \leq \alpha < 1$, if

$$\mathcal{R}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \ (z \in \mathbb{U}). \tag{2}$$

The set of such functions is denoted by $S^*(\alpha)$.

Definition 4 [1]: A function $f \in \mathcal{A}$ is said to be convex of order, $0 \leq \alpha < 1$, if

$$\mathcal{R}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \ (z \in \mathbb{U}). \tag{3}$$

The set of all such functions is denoted by $C(\alpha)$.

It is easily observed from (2) and (3) that (see, [20])

$$f(z) \in C(\alpha) \Leftrightarrow zf'(z) \in S^*(\alpha).$$

Definition 5 [21]: Let $SP(\theta,\alpha)$ denote the subclass of functions $f \in \mathcal{A}$ which satisfy the condition,

$$\Re\left\{e^{i\theta}\left(\frac{zf'(z)}{f(z)} - \alpha\right)\right\} > 0,$$

for some $\left(0 \leq \alpha < 1, \ \frac{-\pi}{2} < \theta < \frac{\pi}{2} \ and \ \ z \in \mathbb{U}\right)$.

Definition 6 [1]: The set $p \in \Omega$ is the set of all functions of the form

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots + c_k z^k + \cdots = 1 + \sum_{k=1}^{\infty} c_k z^k , (z \in \mathbb{U}),$$

that are analytic in the unit disk $\mathbb{U}$ and $\mathcal{R}\{p(z)\} > 0$, any function in Ω is called a function with positive real part in $\mathbb{U}$.

By using the operator $D_l^{m\lambda,}(a,b)f(z)$, we defined two new classes of univalent functions denoted by $V_l^{m,\lambda}(a,b,c,\alpha)$, $V_l^{m,\lambda}(a,b,c,\alpha,\theta)$ as in the following sections.

Definition 7: Let $V_l^{m,\lambda}(a,b,c,\alpha)$ denote the class of $\mathcal{A}$ consisting of functions f satisfying the condition,

$$\Re\left\{1-\frac{2}{c}+\frac{2}{c}\frac{D_l^{m,\lambda}(a+1,b)f(z)}{D_l^{m,\lambda}(a,b)f(z)}\right\}>\alpha,$$

where $(c \neq 0,\ 0 \leq \alpha < 1)$ and $D_l^{m\lambda,}(a,b)f(z)$ is defined in definition 2.

Note that, the parametric values $a=1, b=1, m=0$ *and* $c=2$ and $a=2, b=1, m=0$ *and* $c=1$ we obtain the classes $S^*(\alpha)$ and $C(\alpha)$ respectively see [1].

Definition 8: Let $V_l^{m,\lambda}(a,b,c,\alpha,\theta)$ denote the class of $\mathcal{A}$ consisting of functions f satisfying the condition,

$$\Re\left\{e^{i\theta}\left(1-\frac{2}{c}+\frac{2}{c}\frac{D_l^{m,\lambda}(a+1,b)f(z)}{D_l^{m,\lambda}(a,b)f(z)}-\alpha\right)\right\}>0,$$

for some $\left(0 \leq \alpha < 1,\ \frac{-\pi}{2} < \theta < \frac{\pi}{2}\ \ and\ \ z \in \mathbb{U}\right)$.

Note that, $V_l^{m,\lambda}(a,b,c,\alpha,0) = V_l^{m,\lambda}(a,b,c,\alpha)$ and $V_l^{m,\lambda}(1,1,2,\alpha,\theta) = SP(\theta,\alpha)$ see [2].

Also, we shall need the following lemma in our present investigation.

Lemma 1 [21]: A function $p \in \Omega$ satisfy the following condition $\mathcal{R}\{p(z)\} > 0$ if and only if $p(z) \neq \frac{\Psi-1}{\Psi+1}$ $(\Psi \in \mathbb{C}, |\Psi| = \mathbf{1})$ $\boldsymbol{and}$ $z \in \mathbb{U}$.

3. Coefficient inequalities for the class $V_l^{m,\lambda}(a,b,c,\alpha)$

For our main results, we first derive the following:

Theorem 1: *A* function $f \in \mathcal{A}$ *is in* the class $V_l^{m,\lambda}(a,b,c,\alpha)$ if and only if

$$1+\sum_{k=2}^{\infty}A_k z^{k-1} \neq 0, \tag{4}$$

where

$$A_k = \frac{\Gamma(a+k-1)\Gamma(b)(1+\lambda(k-1)+l)^m}{\Gamma(b+k-1)\Gamma(a+1)(1+l)^m}\left\{\frac{(k-1)+ac(1-\alpha)+(k-1)\psi}{c(1-\alpha)}\right\}a_k,$$

$c \neq 0, 0 \leq \alpha < 1$ and $(z \in \mathbb{U}, b \neq 0,-1,-2,-3,\ldots), \lambda, l \geq 0, m \in \mathbb{Z}$.

Proof: Let us set

$$p(z)=\frac{\left(1-\frac{2}{c}+\frac{2}{c}\frac{D_l^{m,\lambda}(a+1,b)f(z)}{D_l^{m,\lambda}(a,b)f(z)}\right)-\alpha}{1-\alpha}.$$

We find that $p(z)\in\Omega$ and $\Re p(z)>0, z\in\mathbb{U}$.
By applying Lemma 1, we have

$$p(z)=\frac{\left(1-\frac{2}{c}+\frac{2}{c}\frac{D_l^{m,\lambda}(a+1,b)f(z)}{D_l^{m,\lambda}(a,b)f(z)}\right)-\alpha}{1-\alpha}\neq\frac{\psi-1}{\psi+1},\qquad (\psi\in\mathbb{C},|\psi|=1,z\in\mathbb{U}),$$

which readily yields

$$(\psi+1)D_l^{m,\lambda}(a+1,b)f(z)+\left[c(1-\alpha)-(1+\psi)\right]D_l^{m,\lambda}(a,b)f(z)\neq 0$$

$$f(z)\in V_l^{m,\lambda}(a,b,c,\alpha),\psi\in\mathbb{C},|\psi|=1,z\in\mathbb{U}.$$

Thus we find that

$$(\psi+1)\left(z+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m\frac{(a+1)_{k-1}}{(b)_{k-1}}a_kz^k\right)+\left[c(1-\alpha)-(1+\psi)\right]$$

$$\left(z+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m\frac{(a)_{k-1}}{(b)_{k-1}}a_kz^k\right)\neq 0,$$

which readily gives

$$c(1-\alpha)z\,(1+\sum_{k=2}^{\infty}\frac{\Gamma(a+k-1)\Gamma(b)(1+\lambda(k-1)+l)^m}{\Gamma(b+k-1)\Gamma(a+1)(1+l)^m}$$

$$\left\{\frac{(k-1)+ac(1-\alpha)+(k-1)\psi}{c(1-\alpha)}\right\}a_kz^{k-1})\neq 0,\qquad (5)$$

where $(\psi\in\mathbb{C},|\psi|=1,z\in\mathbb{U})$.
Dividing both sides of (5) by $c(1-\alpha)z$ we obtain,

$$1+\sum_{k=2}^{\infty}\frac{\Gamma(a+k-1)\Gamma(b)(1+\lambda(k-1)+l)^m}{\Gamma(b+k-1)\Gamma(a+1)(1+l)^m}\left\{\frac{(k-1)+ac(1-\alpha)+(k-1)\psi}{c(1-\alpha)}\right\}a_kz^{k-1}\neq 0,$$

where $(\psi\in\mathbb{C},|\psi|=1,z\in\mathbb{U})$, which complete the proof of Theorem 1.

Remark 1: The parametric substitutions,

1. $a=\delta+1, b=1$ and $m=0$ yields Lemma1.4, in [22].
2. $a=1, b=1, c=2$ and $m=0$ yields Lemma 2, in [21].

In view of Theorem 2, we state and proof the following,

Theorem 2: If $f \in \mathcal{A}$ satisfies the following condition:

$$\sum_{k=2}^{\infty} \left| \sum_{n=1}^{k} \left[\sum_{j=1}^{n} \frac{\Gamma(a+j-1)\Gamma(b)(1+\lambda(j-1)+l)^m}{\Gamma(b+j-1)\Gamma(a+1)(1+l)^m} (-1)^{n-j} \times [(j-1)+ac(1-\alpha)] \begin{pmatrix} \beta \\ n-j \end{pmatrix} a_j \right] \begin{pmatrix} \gamma \\ k-n \end{pmatrix} \right| + \left| \sum_{n=1}^{\infty} \left[\sum_{j=1}^{n} \frac{\Gamma(a+j-1)\Gamma(b)(1+\lambda(j-1)+l)^m}{\Gamma(b+j-1)\Gamma(a+1)(1+l)^m} (-1)^{n-j} (j-1) \begin{pmatrix} \beta \\ n-j \end{pmatrix} a_j \right] \begin{pmatrix} \gamma \\ k-n \end{pmatrix} \right| \le c(1-\alpha),$$

for some $\alpha (0 \le \alpha < 1)$, $\beta, \gamma \in \mathbb{R}$, then $f(z) \in V_l^{m,\lambda}(a,b,c,\alpha)$.

Proof: First, we note that

$$(1-z)^{\beta} \neq 0, \quad (1+z)^{\gamma} \neq 0, \quad (\beta, \gamma \in \mathbb{R}, z \in \mathbb{U}).$$

Hence, if the following inequality

$$\left(1+\sum_{k=2}^{\infty} A_k z^{k-1}\right)(1-z)^{\beta}(1+z)^{\gamma} \neq 0, \quad (\beta, \gamma \in \mathbb{R}, z \in \mathbb{U}),$$

holds true, then we have that

$$1+\sum_{k=2}^{\infty} A_k z^{k-1} \neq 0,$$

which is the relation (5) of Theorem 1, we get

$$\left(1+\sum_{k=2}^{\infty} A_k z^{k-1}\right)\left(\sum_{k=0}^{\infty} (-1)^k b_k z^k\right)\left(\sum_{k=0}^{\infty} c_k z^k\right) \neq 0, \tag{6}$$

where, for convenience,

$$b_k = \begin{pmatrix} \beta \\ k \end{pmatrix} \text{ and } c_k = \begin{pmatrix} \gamma \\ k \end{pmatrix}.$$

Considering the Cauchy product of the first two factors of (6) can be rewritten as follows:

$$\left(1+\sum_{k=2}^{\infty} B_k z^{k-1}\right)\left(\sum_{k=0}^{\infty} c_k z^k\right) \neq 0, \tag{7}$$

where $B_k = \sum_{j=1}^{k} (-1)^{k-j} A_j b_{k-j}$.

Furthermore, by applying the same Cauchy product method in (7), we obtain

$$1+\sum_{k=2}^{\infty}\left(\sum_{n=1}^{k}B_n c_{k-n}\right)z^{k-1}\neq 0,\quad (z\in\mathbb{U}),$$

or written equivalently as

$$1+\sum_{k=2}^{\infty}\left(\sum_{n=1}^{k}\left[\sum_{j=1}^{n}(-1)^{n-j}A_j b_{n-j}\right]c_{k-n}\right)z^{k-1}\neq 0.$$

That is if $f\in\mathcal{A}$ satisfies the following inequality

$$\sum_{k=2}^{\infty}\left|\sum_{n=1}^{k}\left[\sum_{j=1}^{n}(-1)^{n-j}A_j b_{n-j}\right]c_{n-j}\right|\leq 1,$$

that is, if

$$\frac{1}{c(1-\alpha)}\sum_{k=2}^{\infty}|\sum_{n=1}^{k}[\sum_{j=1}^{n}\frac{\Gamma(a+j-1)\Gamma(b)(1+\lambda(j-1)+l)^m}{\Gamma(b+j-1)\Gamma(a+1)(1+l)^m}(-1)^{n-j}$$

$$\times[(j-1)+ac(1-\alpha)]a_j b_{n-j}]c_{k-n}|+|\psi||\sum_{n=1}^{k}[\sum_{j=1}^{n}\frac{\Gamma(a+j-1)\Gamma(b)(1+\lambda(j-1)+l)^m}{\Gamma(b+j-1)\Gamma(a+1)(1+l)^m}$$

$$(j-1)b_{n-j}a_j]c_{k-n}|\leq 1,$$

where $(0\leq\alpha<1,\psi\in\mathbb{C},|\psi|=1,z\in\mathbb{U})$, then $f(z)\in V_l^{m,\lambda}(a,b,c,\alpha)$.
This completes the proof of Theorem 2.

Remark 2: In the hypothesis of Theorem 2, for the parametric values

1. $a=\delta+1, b=1$ and $m=0$ yields Theorem 2.1, in [21].
2. $a=1, b=1, c=2$ and $m=0$ yields Theorem 1, in [22].
3. $a=2, b=1, c=1$ and $m=0$ yields Theorem 2, in [22].

By specializing on the parameters $a,b,c,\alpha,\beta,\gamma$ and λ and in Theorem 2, we can deduce the following interesting corollaries.

For $a=b=1, c=2$ and $\alpha=m=0$, we have

Corollary 1: *If $f\in\mathcal{A}$ satisfies*

$$\sum_{k=2}^{\infty}\left|\sum_{n=1}^{k}\left[\sum_{j=1}^{n}(-1)^{n-j}(j+1)\binom{\beta}{n-j}a_j\right]\binom{\gamma}{k-n}\right|+$$

$$\left|\sum_{n=1}^{\infty}\left[\sum_{j=1}^{n}(-1)^{n-j}(j-1)\binom{\beta}{n-j}a_j\right]\binom{\gamma}{k-n}\right|\leq 2,$$

for some $\beta,\gamma\in\mathbb{R}$, then $f\in S^*$.

For $a = 2, b = c = 1$ and $\alpha = m = 0$, we have,

Corollary 2: *If $f \in \mathcal{A}$ satisfies*

$$\sum_{k=2}^{\infty}\left|\sum_{n=1}^{k}\left[\sum_{j=1}^{n}(-1)^{n-j}\, j\,(j+1)\binom{\beta}{n-j}a_j\right]\binom{\gamma}{k-n}\right|+$$
$$\left|\sum_{n=1}^{\infty}\left[\sum_{j=1}^{n}(-1)^{n-j}\, j\,(j-1)\binom{\beta}{n-j}a_j\right]\binom{\gamma}{k-n}\right|\le 2,$$

for some $\beta, \gamma \in \mathbb{R}$, then $f \in C$.

For $a = b = 1, c = 2$ and $\beta = \gamma = m = 0$, we have,

Corollary 3 [23]: *If $f \in \mathcal{A}$ satisfies*

$$\sum_{k=2}^{\infty}(k-\alpha)\,|a_k| \le 1-\alpha,$$

for some $0 \le \alpha < 1$, then $f \in S^*(\alpha)$.

For $a = 2, b = 1, c = 1$ and $\beta = \gamma = m = 0$, we have

Corollary 4 [23]: *If $f \in \mathcal{A}$ satisfies*

$$\sum_{k=2}^{\infty}k\,(k-\alpha)\,|a_k| \le 1-\alpha,$$

for some $0 \le \alpha < 1$, then $f \in C(\alpha)$.

For $a = b = 1, c = 2$ and $\alpha = \beta = \gamma = m = 0$, we have

Corollary 5 [1]: *If $f \in \mathcal{A}$ satisfies*

$$\sum_{k=2}^{\infty}k\,|a_k| \le 1,$$

then $f \in S^*$.

For $a = 2, b = 1, c = 2$ and $\alpha = \beta = \gamma = m = 0$, we have

Corollary 6 [1]: *If $f \in \mathcal{A}$ satisfies*

$$\sum_{k=2}^{\infty}k^2\,|a_k| \le 1,$$

then $f \in C$.

4. Coefficient Inequality for Functions in the Class $V_l^{m,\lambda}(a,b,c,\alpha,\theta)$

Theorem 3: A function $f \in \mathcal{A}$ is in the class $V_l^{m,\lambda}(a,b,c,\alpha,\theta)$ if and only if

$$1+\sum_{k=2}^{\infty} L_k z^{k-1} \neq 0, \tag{8}$$

where

$$L_k=\frac{\Gamma(a+k-1)\Gamma(b)(1+\lambda(k-1)+l)^m}{\Gamma(b+k-1)\Gamma(a+1)(1+l)^m}\times\left\{\frac{(k-1)+ac(1-\alpha)e^{-i\theta}\cos\theta+(k-1)\psi}{c(1-\alpha)e^{-i\theta}\cos\theta}\right\}a_k,$$

and

$c\neq 0, 0\leq\alpha<1$ and $(z\in\mathbb{U}, b\neq 0,-1,-2,-3,\ldots), \lambda\geq 0, m\in\mathbb{Z}, l\geq 0.$

Proof: Let us set

$$p(z)=\frac{e^{i\theta}\left(1-\frac{2}{c}+\frac{2}{c}\frac{D_l^{m,\lambda}(a+1,b)f(z)}{D_l^{m,\lambda}(a,b)f(z)}-\alpha\right)-i(1-\alpha)\sin\theta}{(1-\alpha)\cos\theta}.$$

We find that $p(z)\in\Omega$ and $\Re p(z)>0, z\in\mathbb{U}$.

From Lemma 1, it follows that,

$$p(z)=\frac{e^{i\theta}\left(1-\frac{2}{c}+\frac{2}{c}\frac{D_l^{m,\lambda}(a+1,b)f(z)}{D_l^{m,\lambda}(a,b)f(z)}-\alpha\right)-i(1-\alpha)\sin\theta}{(1-\alpha)\cos\theta}\neq\frac{\psi-1}{\psi+1}, \qquad (\psi\in\mathbb{C}, |\psi|=1, z\in\mathbb{U}),$$

which readily yields

$$(\psi+1)\left\{e^{i\theta}[2D_l^{m,\lambda}(a+1,b)f(z)-[c\alpha+2-c]e^{i\theta}D_l^{m,\lambda}(a,b)f(z)]-i(1-\alpha)cD_l^{m,\lambda}(a,b)f(z)\sin\theta\right\}$$

$$\neq(\psi-1)(1-\alpha)cD_l^{m,\lambda}(a,b)f(z)\cos\theta,$$

where $\psi\in\mathbb{C}, |\psi|=1, z\in\mathbb{U}$.

Or equivalently

$$\begin{gathered}2(\psi+1)e^{i\theta}D_l^{m,\lambda}(a+1,b)f(z)-[c\alpha+2-c]e^{i\theta}D_l^{m,\lambda}(a,b)f(z)-\psi[c\alpha+2-c]e^{i\theta}D_l^{m,\lambda}(a,b)f(z)\\-i(1-\alpha)cD_l^{m,\lambda}(a,b)f(z)\sin\theta-i\psi(1-\alpha)cD_l^{m,\lambda}(a,b)f(z)\sin\theta\end{gathered}$$

$$\neq\psi(1-\alpha)cD_l^{m,\lambda}(a,b)f(z)\cos\theta-(1-\alpha)cD_l^{m,\lambda}(a,b)f(z)\cos\theta,$$

where $\psi\in\mathbb{C}, |\psi|=1, z\in\mathbb{U}$.

Further simplification yields,

$$\begin{gathered}2(\psi+1)e^{i\theta}D_l^{m,\lambda}(a+1,b)f(z)-[c\alpha+2-c]e^{i\theta}D_l^{m,\lambda}(a,b)f(z)-\psi[c\alpha+2-c]e^{i\theta}D_l^{m,\lambda}(a,b)f(z)\\-\psi(1-\alpha)ce^{i\theta}D_l^{m,\lambda}(a,b)f(z)+(1-\alpha)ce^{i\theta}D_l^{m,\lambda}(a,b)f(z)\neq 0,\end{gathered}$$

where $\psi\in\mathbb{C}, |\psi|=1, z\in\mathbb{U}$,

then,

$$2(\psi+1)e^{i\theta}D_l^{m,\lambda}(a+1,b)f(z)+[ce^{-i\theta}-(2\psi-c+2)e^{i\theta}-2\alpha c\cos\theta]D_l^{m,\lambda}(a,b)f(z)\neq 0,$$

where $\psi \in \mathbb{C}, |\psi| = 1, z \in \mathbb{U}$.

Since $a_0 = a_1 - 1 = 0$, we have

$$2c(1-\alpha)z\cos\theta(1+\sum_{k=2}^{\infty}\frac{\Gamma(a+k-1)\Gamma(b)(1+\lambda(k-1)+l)^m}{\Gamma(b+k-1)\Gamma(a+1)(1+l)^m}$$

$$\times\left\{\frac{2(k-1)(\psi+1)+a(ce^{-2i\theta}-2\alpha ce^{-i\theta}\cos\theta+c)}{2c(1-\alpha)e^{-i\theta}\cos\theta}\right\}a_k z^{k-1}) \neq 0, \qquad (9)$$

where $\psi \in \mathbb{C}, |\psi| = 1, z \in \mathbb{U}$.

Finally, upon dividing both sides of (9) by $2c(1-\alpha)z\cos\theta$, and noting that

$$e^{-2i\theta} = -1 + 2e^{-i\theta}\cos\theta,$$

we get,

$$1+\sum_{k=2}^{\infty}\frac{\Gamma(a+k-1)\Gamma(b)(1+\lambda(k-1)+l)^m}{\Gamma(b+k-1)\Gamma(a+1)(1+l)^m}$$

$$\times\left\{\frac{(k-1)+ac(1-\alpha)e^{-i\theta}\cos\theta+(k-1)\psi}{c(1-\alpha)e^{-i\theta}\cos\theta}\right\}a_k \neq 0,$$

$c \neq 0, 0 \le \alpha < 1$ and $(z \in \mathbb{U}, b \neq 0, -1, -2, -3, \ldots), \lambda \ge 0, m \in \mathbb{Z}, l \ge 0$.

Which complete the proof of Theorem 3.

Remark 3: The parametric substitutions,

1. $a = \delta + 1, b = 1$ and $\theta = m = 0$ yields Lemma 3.1, in [3].
2. $a = 1, b = 1, c = 2$ an $\theta = m = 0$ yields Lemma3, in [2].

Theorem 4: If $f \in \mathcal{A}$ satisfies the following condition:

$$\sum_{k=2}^{\infty}\left|\sum_{n=1}^{k}\left[\sum_{j=1}^{n}\frac{\Gamma(a+j-1)\Gamma(b)(1+\lambda(j-1)+l)^m}{\Gamma(b+j-1)\Gamma(a+1)(1+l)^m}(-1)^{n-j}\times[2(j-1)+ac(1-\alpha)(1+e^{-2i\theta})]\right.\right.$$

$$\left.\left.\binom{\beta}{n-j}a_j\right]\binom{\gamma}{k-n}\right|+\left|\sum_{n=1}^{\infty}\left[\sum_{j=1}^{n}\frac{\Gamma(a+j-1)\Gamma(b)(1+\lambda(j-1)+l)^m}{\Gamma(b+j-1)\Gamma(a+1)(1+l)^m}(-1)^{n-j}2(j-1)\right.\right.$$

$$\left.\left.\binom{\beta}{n-j}a_j\right]\binom{\gamma}{k-n}\right| \le 2c(1-\alpha)\cos\theta,$$

for some $\alpha(0 \le \alpha < 1)$, $-\pi/2 < \theta < \pi/2, \beta, \gamma \in \mathbb{R}$,

then $f(z) \in V_l^{m,\lambda}(a,b,c,\alpha,\theta)$.

Proof: Applying the same method as in the proof of Theorem 2, we observe that f is in the class $V_l^{m,\lambda}(a,b,c,\alpha,\theta)$ if,

$$\sum_{k=2}^{\infty}\left|\sum_{n=1}^{k}\left[\sum_{j=1}^{n}(-1)^{n-j}L_j b_{n-j}\right]c_{k-n}\right|\leq 1, \tag{10}$$

where, for convenience, $b_k=\begin{pmatrix}\beta\\k\end{pmatrix}$ and $c_k=\begin{pmatrix}\gamma\\k\end{pmatrix}$,

the coefficients L_k being given as in Lemma 1.

From (10) it follows that,

$$\frac{1}{|c(1-\alpha)e^{-i\theta}\cos\theta|}\sum_{k=2}^{\infty}|\sum_{n=1}^{k}[\sum_{j=1}^{n}\frac{\Gamma(a+j-1)\Gamma(b)(1+\lambda(j-1)+l)^m}{\Gamma(b+j-1)\Gamma(a+1)(1+l)^m}(-1)^{n-j}$$

$$\times[(j-1)+ac(1-\alpha)e^{-i\theta}\cos\theta+\psi(j-1)]a_j b_{n-j})c_{k-n}|\leq\frac{1}{2c(1-\alpha)\cos\theta}\sum_{k=2}^{\infty}|\sum_{n=1}^{k}[\sum_{j=1}^{n}$$

$$\frac{\Gamma(a+j-1)\Gamma(b)(1+\lambda(j-1)+l)^m}{\Gamma(b+j-1)\Gamma(a+1)(1+l)^m}(-1)^{n-j}\times[2(j-1)+ac(1-\alpha)(1+e^{-2i\theta})]a_j b_{n-j}]c_{k-n}|$$

$$+|\psi||\sum_{n=1}^{k}[\sum_{j=1}^{n}\frac{\Gamma(a+j-1)\Gamma(b)(1+\lambda(j-1)+l)^m}{\Gamma(b+j-1)\Gamma(a+1)(1+l)^m}2(j-1)b_{n-j}a_j]c_{k-n}|\leq 1,$$

where $(0\leq\alpha<1,-\pi/2<\theta<\pi/2,\psi\in\mathbb{C},|\psi|=1,z\in\mathbb{U})$,

which implies that $f(z)\in V_l^{m,\lambda}(a,b,c,\alpha,\theta)$. This completes the proof.

Remark 4: For $\theta=0$, Theorem *3* implies Theorem *2.*

Also for the parametric substitutions,

1. $a=\delta+1, b=1$ and $m=0$ Theorem3 yields Theorem 3.3, in [21].
2. $a=1, b=1, c=2$ and $m=0$ Theorem 3 yields Theorem 3, in [22].

For $a=b=1, c=2$ and $\alpha=m=0$, we have

Corollary 7: If $f\in\mathcal{A}$ satisfies

$$\sum_{k=2}^{\infty}\left|\sum_{n=1}^{k}\left[\sum_{j=1}^{n}(-1)^{n-j}(j+e^{-2i\theta})]\begin{pmatrix}\beta\\n-j\end{pmatrix}a_j\right]\begin{pmatrix}\gamma\\k-n\end{pmatrix}\right|+$$

$$\left|\sum_{n=1}^{\infty}\left[\sum_{j=1}^{n}(-1)^{n-j}(j-1)\begin{pmatrix}\beta\\n-j\end{pmatrix}a_j\right]\begin{pmatrix}\gamma\\k-n\end{pmatrix}\right|\leq 2\cos\theta,$$

for some $\alpha(0\leq\alpha<1)$, $-\pi/2<\theta<\pi/2,\beta,\gamma\in\mathbb{R}$, then $f(z)\in SP(\theta,0)$.

Conclusion

This study has showed and proved some sufficient conditions for functions defined by a linear operator. The work is also completed by remarking that the presented results can be used to investigate some properties of subclasses of analytic functions satisfying certain coefficient inequality.

References

[1] W. Goodman, "Univalent functions," Polygonal Publishing House, Washington, Vol. I and II, 1983.

[2] W. Kaplan, "Close-to-convex schlicht functions," Michigan Mathematical Journal, Vol. 1(2), 169–185,1952.

[3] S. Salagean, "Subclasses of univalent functions," Lecture notes in math. (Springer-Verlag), 1013, 362-372, 1983.

[4] A. Catas, "On a Certain Differential Sandwich Theorem Associated with a New Generalized Derivative Operator," General Mathematics, Vol. 4, 83-95, 2009.

[5] S. Ruscheweyh, "New criteria for univalent functions," Proc. Amer. Math. Soc. Vol. 49, 109-115, 1975.

[6] A.A. Amer, M. Darus, N.M. Alabbar, "Properties for Generalized Starlike and Convex Functions of Order α," Fezzan University Scientific Journal, Vol. 3(1), 423-429, 2024.

[7] N.M Alabbar, M. Darus. A.A. Amer, "Coefficient Inequality and Coefficient Bounds for a New Subclass of Bazilevic Functions," Journal of Humanitarian and Applied Sciences, Vol. 8, 496-506, 2023.

[8] N.M Alabbar, A.A. Amer, "Properties of Generalized Derivative Operator to A Certain Subclass of Analytic Functions with Negative Coefficients," Global Libyan Journal Vol. 2, 2017.

[9] N.M. Mustafa, M. Darus, "Some properties of a subclass of analytic function defined by a generalized Srivastava and Attiya operator," Facta Universitatis (NIS), Vol. 27(3), 309-320, 2012.

[10] F. Al-Oboudi, "On univalent functions defined by a generalised Salagean Operator," Int, J. Math. Math. Sci. Vol. 27, 1429-1436,2004.

[11] A.A. Amer, M. Darus, N.M. Alabbar, "Necessary conditions for the generalized derivative operator in classes of univalent functions," Academy journal for Basic and Applied Sciences, Vol. 6(2), 1-15, August 2024.

[12] E. Shmella, A.A. Amer, "Estimation of the bounds of univalent functional of coefficients apply the subordination method," The Academic Open Journal of Applied And Human Sciences, Vol. 5(1), 2709-3344, 2023.

[13] E. Shmella, A.A. Amer, "Some properties of differential subordination for the subordination class with the generalized derivative operator," Bani Waleed University Journal of Humanities and Applied Sciences, Vol. 2, 390-400, 2024.

[14] A.A. Amer, M. Darus, “On some properties for new generalized derivative operator,” Jordan Journal of Mathematics and Statistics, Vol. 4, 91-101, 2011.

[15] F.A. Abufares, A.A. Amer, “Certain applications of analytic functions associated in complex BB differential equations,” Journal of the Faculty of Education Tripoli, 19(1), 264-274, 2024.

[16] A.A. Amer, “Second Hankel determinant for new subclass defined by a linear operator,” Springer International Publishing Switzerland, Chapter 6, 2016.

[17] A.A. Amer. M. Darus, “Some Properties of the Class of Univalent Functions with Negative Coefficients” Applied Mathematics, Vol. 3, 1851-1856, 2012.

[18] C. Carlson, B. Shaffer, “Starlike and prestarlike hypergeometric functions,” SIAM J. Math. Anal. Vol. 15(4), 737-745, 1984.

[19] S. Owa, H. M. Srivastava, “Univalent and starlike generalized hypergeometric functions,” Can. J. Math, Vol. 39, 1057-1077, 1987.

[20] H.M. Srivastava, S. Owa, (Editors), “Current Topics in Analytic Function Theory,” World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1992.

[21] T. Hayami, S. Owa, H.M. Srivastava, “Coefficient inequalities for certain classes of analytic and univalent functions,” J. Ineq. Pure Appl. Math. Vol. 8(4), 95, 1-10, 2007.

[22] S. Latha, O. Karthiyayini, “Certain class of analytic and univalent functions involving the Ruscheweyh derivative operator,” Int. Journal of Math. Analysis, Vol. 3(33), 1633–1644, 2009.

[23] H.M. Srivastava, S. Owa, S. Chatterjea, “A note on certain classes of starlike functions,” Rendiconti del Seminario Matematico della University di Padova, tome Vol. 77, 115-124, 1987.