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A Numerical Study for Solving the Logistic Differential Equation with Caputo-Fractional Derivative and Time Delay

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الملخص— المعادلات التفاضلية ذات الرتب الكسرية هي تقنية فعالة لاختبار النظريات، والتحقق من صحة النتائج التجريبية، وتمثيل ديناميكيات الأنظمة المعقدة. هناك تعميمات مختلفة لمعادلات التفاضلية اللوجستية تم أخذها في الاعتبار. نظرًا للطبيعة الخطية والغير الخطية لمعادلات التفاضلية اللوجستية، يتم إدخال طرق مختلفة للحصول على الحل. في هذه الورقة نقدم تعميمًا لمعادلة التفاضلية اللوجستية، وهو ما يضمن ويُحسن هذا التعميم كحالات خاصة. يهدف هذا العمل إلى دراسة طريقة تحويل الزاكي للاضطراب المثلي (HPETM) لحل المعادلات التفاضلية اللوجستية الغير الخطية بمشتقة كابوتو الكسري والتأخير الزمني. وفي الختام، أظهرت النتائج، أن هذه الطريقة سهلة التنفيذ، متقاربة وفعالة لحل هذا النوع من المعادلات التفاضلية ذات الرتب الكسرية.

الكلمات المفتاحية— المعادلة التفاضلية اللوجستية ذات الرتبة الكسرية والتأخير الزمني، المؤثر الكسري لكابوتو، تحويل الزاكي، طريقة الاضطراب المثلي.

Abstract—Fractional differential equations is an effective technique to test theories, validate experimental findings, and represent the dynamics of complex systems. There are different generalizations of the logistic differential equations that have been considered. Since the linear and nonlinear nature of the logistic differential equation, various methods are input to obtain a solution. In this paper we present a generalization of the logistic differential equation, which guarantees and improves this generalization as special cases. This work aims to study the Elzaki homotopy perturbation method (EHPM) to solve the nonlinear logistic differential equation with Caputo derivative and the time delay. In conclusion, the numerical results showed that this method is easy to implement, convergent, and effective for solving this type of fractional differential equations.

Keywords—Fractional-order; Caputo derivative; Elzaki transform; Homotopy perturbation method

1. Introduction

Fractional calculus has been employed in many different domains, such as engineering, physics, applied mathematics, biology, and mechanics. It has recently been used to demonstrate facts [1,2]. For the majority of the function's history, fractional derivatives are more interesting. The Riemann-Liouville and Caputo definitions are just two examples of the many functional derivatives that are provided in the literature [3,4]. Pierre Verhulst initially examined the following population growth model [5], ${dN}/{d\tau} = \varepsilon N(1 - {N}/{K})$, where $N(\tau)$ is population at time τ, and $\varepsilon > 0$ is Malthusian parameter describing growth rate and K is carrying capacity. Defining $\zeta = N/K$ yields the differential equation that follows $d\zeta/d\tau = \varepsilon\zeta(1-\zeta)$, which is called as logistic equation. The logistic equation of fractional order has been discussed in the literature [6-8] as follows ${}_0^C D_\tau^\gamma \zeta(\tau) = \varepsilon\zeta(\tau)(1-\zeta(\tau))$, where Caputo fractional derivative of order $0<\gamma\le 1$. In [9,10], a broad and more logistic model ${}_0^C D_\tau^\gamma \zeta(\tau) = \varepsilon\zeta(\tau)(1-\zeta(\rho\tau))$, which involves the Caputo–Fabrizio fractional derivative and proportional time delay, $0<\rho<1$, where $\rho\in[0,1]$. Recently, the use of fractional differential equations in non-linear dynamics has received significant attention in recent literature. Researchers have shown strong interest in solving nonlinear fractional differential equations with time delay and applying them to logistic models involving the Caputo derivative, which is one of the most important and intriguing topics in the field [11–16]. The Homotopy Perturbation Method (HPM) [17,18] is a general technique for solving fractional ordinary differential equations (FODEs). This approach was first presented by He [19]. The HPM is a combination of homotopy and perturbation methods. Several studies [20-23] have demonstrated that the Elzaki transform may be used for partial differential equations, ordinary differential equations, systems of ordinary and partial differential equations, and integral equations. This study presents a logistic model with the Caputo-fractional derivative and time delay. The model incorporates and improves some existing models as special cases. The study demonstrates a hybrid homotopy perturbation Elzaki Transform method (HPETM) approach for solving nonlinear delay fractional differential equations, with an application to the logistic model. In order to demonstrate that the approximate analytical method produces a reliable solution, the paper presents some numerical approximations using MATLAB 2013 and arbitrary parameter choices. It also illustrates the dynamics of the logistic differential equation, which are determined by the Caputo-fractional derivative and time delay.

2. Preliminary to Illustrate Key Concepts and Definitions

We begin by defining the Elzaki transform and the Caputo-fractional derivative, which will be used throughout this work.

Definition 1 [3-4]: The Riemann-Liouville fractional integral of order $\gamma>0$, of a function $\zeta\in C_\sigma, \sigma\ge -1$ is defined as:

$$J^\gamma \zeta(\tau) = \frac{1}{\Gamma(\gamma)}\int_0^\tau (\tau-t)^{\gamma-1}\zeta(t)\,dt. \tag{1}$$

Definition 2 [3-4]: Let $\zeta\in C_m^n, n\in\mathbb{N}\cup\{0\}$. The Caputo fractional derivative of ζ in the Caputo sense is defined as follows:

$${}_0^C D_\tau^\gamma \zeta(\tau) = \begin{cases} \dfrac{1}{\Gamma(n-\gamma)}\displaystyle\int_0^\tau (\tau-t)^{n-\gamma-1}\zeta^{(n)}(t)\;dt, & n-1<\gamma\le n, \\ D_\tau^\gamma \zeta(\tau), & \gamma = n. \end{cases} \tag{2}$$

Definition 3 [20]: Elzaki transform over a group of functions A is defined

$$A = \left\{\zeta(\tau)/\exists\ \mathbb{N},\ t_1,\ t_2>0,\ |\omega(\tau)| < N\ E\left(\frac{|\tau|}{t_i}\right), if\ \tau\in(-1)^i\times 0,\infty)\right\}. \tag{3}$$

By integration that follows:

$$E\left[\zeta(\tau)\right](\vartheta) = \int_0^\infty e^{-\tau/\vartheta}\,\zeta(\tau)\;d\tau, \quad \tau>0, \tag{4}$$

where ϑ is a Elzaki transform parameter.

The following are some of the Elzaki transform's special properties:

$$S[1]=\vartheta^2, \quad S\left[\frac{\tau^m}{\Gamma(m+1)}\right]=\vartheta^{m+2}. \tag{5}$$

Definition 4 [20]: The Elzaki transform for the Caputo fractional derivative for $0<\gamma\le 1$ is defined as:

$$S\left[{}_0^C D_\tau^\gamma \zeta(\tau)\right](\vartheta)=\frac{E[\zeta(\tau)]}{\vartheta^\gamma}-\sum_{k=0}^{n-1}\vartheta^{k-\gamma+2}\zeta^{(k)}(0). \tag{6}$$

3. Analysis of Homotopy Perturbation Elzaki Transform Method (HPETM), Uniqueness and Convergence

3.1 Analysis of Homotopy Perturbation Elzaki Transform Method (HPETM)

In Caputo operator sense, consider the following, a general nonlinear fractional differential equation with initial condition:

$${}_0^C D_\tau^\gamma \zeta(\tau)+L\,\zeta(\tau)+N\,\zeta(\tau)=g(\tau), \quad 0<\gamma\le 1. \tag{7}$$

subject to the initial condition

$$\zeta(0)=\zeta_0(\tau), \tag{8}$$

where ${}_0^C D_\tau^\gamma$ is the γ order fractional Caputo derivative, ζ is the unknown function, L and N are linear and nonlinear operators, and g is the source term.

Applying the Elzaki transform to both sides of the Equation (7)

$$E\left[{}_0^C D_\tau^\gamma \zeta(\tau)+L\,\zeta(\tau)+N\,\zeta(\tau)\right]=E[g(\tau)], \tag{9}$$

Using Definition 2.4 and Equation (8) in Equation (9), gives

$$E[\zeta(\tau)]=\zeta(0)+\vartheta^\gamma E\left[g(\tau)-L\,\zeta(\tau)-N\,\zeta(\tau)\right]. \tag{10}$$

Using the inverse Elzaki transform to both sides of the Equation (10)

$$\zeta(\tau)=E^{-1}\left[\left[\zeta(0)+\vartheta^\gamma E\left[g(\tau)-L\,\zeta(\tau)-N\,\zeta(\tau)\right]\right.\right.. \tag{11}$$

Now we apply the HPM

$$\zeta(\tau)=\sum_{m=0}^{\infty}q^m\zeta_m(\tau). \tag{12}$$

The nonlinear term is decomposed as:

$$N\,\zeta(\tau)=\sum_{m=0}^{\infty}q^m B_m. \tag{13}$$

For some Adomian's polynomials B_m that are given by [24]

$$B_m=\frac{1}{m!}\frac{d^m}{dq^m}\left[N\left(\sum_{i=0}^{\infty}q^i\zeta_i\right)\right]_{q=0}, \quad m=0,1,2,\ldots. \tag{14}$$

Substituting Equation (12) and Equation (13) in Equation (11), we get,

$$\sum_{m=0}^{\infty} q^{m} \zeta_{m+1}(\tau) = \zeta(0) + q\left(E^{-1}\left[\left[\vartheta^{\gamma} E\left[g(\tau) - L\left(\sum_{m=0}^{\infty} q^{m} \zeta_{m}(\tau)\right) - N\left(\sum_{m=0}^{\infty} q^{m} B_{m}\right)\right]\right]\right]. \quad (15)$$

Equating the terms with identical powers of q yields the following equations:

$$\begin{aligned} q^{0} &: \zeta_0(\tau) = \zeta(0), \\ q^{1} &: \zeta_1(\tau) = E^{-1}\left[\vartheta^{\gamma} E\left[g(\tau) - L\,\zeta_0(\tau) - NB_0\right]\right], \\ q^{2} &: \zeta_2(\tau) = E^{-1}\left[\vartheta^{\gamma} E\left[g(\tau) - L\,\zeta_1(\tau) - NB_1\right]\right], \\ &\vdots \\ q^{m} &: \zeta_{m+1}(\tau) = E^{-1}\left[\vartheta^{\gamma} E\left[g(\tau) - L\,\zeta_m(\tau) - NB_m\right]\right]. \end{aligned} \quad (16)$$

By following the same procedure, the remaining components $\zeta(\tau)$ may be obtained, and the series solution is thus completely defined. Using truncated series, we approximate the analytical solution as follows:

$$\zeta(\tau) = \lim_{M \to \infty} \sum_{m=0}^{M} q^{m} \zeta_m(\tau). \quad (17)$$

3.2 Uniqueness Theorem

Let $B = \lambda(\Omega, \mathbb{R})$ denote the Banach space of all the continuous functions ζ on $\Omega = \mathbb{R} \times [0, T]$ with the norm $norm\,\|\zeta(\tau)\| = \max_{\tau \in \Omega} |\zeta(\tau)|.$ (18)

Theorem 3.1: Suppose that L and N are also Lipschitzion with $|L\zeta - L\xi| < \lambda_1 |\zeta - \xi|$ and $|N\zeta - N\xi| < \lambda_2 |\zeta - \xi|$ where λ_1 and λ_2 are Lipschitz constants. ζ and ξ are two different function values. Then the solution (17) is a unique solution for equation (7).

$$0 < (\lambda_1 + \lambda_2)\vartheta^{\gamma+2} \le 1$$

Proof: At the beginning, we define the operator $f : B \to B$ where

$$\zeta_{m+1}(\tau) = \zeta_m(\tau) + E^{-1}\left[\vartheta^{\gamma} E\left[g(\tau) - L\,\zeta_m(\tau) - N\,\zeta_m(\tau)\right]\right]. \quad (19)$$

In order to investigate the existence and uniqueness of the solution to equation (7), we use Banach fixed point theorem. For this, let $\zeta, \xi \in B$, we have

$$\begin{aligned} \|f\,\zeta - f\,\xi\| &= \max_{\tau \in \Omega} \left| E^{-1}\left[\vartheta^{\gamma} E\left[L\,\zeta + N\,\zeta\right]\right] - E^{-1}\left[\vartheta^{\gamma} E\left[L\,\xi + N\,\xi\right]\right] \right\|, \\ &= \max_{\tau \in \Omega} \left| E^{-1}\left[\vartheta^{\gamma} E\left[L\,\zeta - L\,\xi\right]\right] + E^{-1}\left[\vartheta^{\gamma} E\left[N\,\zeta - N\,\xi\right]\right] \right\|, \\ &\le \max_{\tau \in \Omega} \left| \lambda_1 E^{-1}\left[\vartheta^{\gamma} E\left[|\zeta - \xi|\right]\right] + \lambda_2 E^{-1}\left[\vartheta^{\gamma} E\left[|\zeta - \xi|\right]\right] \right\|, \\ &\le \max_{\tau \in \Omega} \left| (\lambda_1 + \lambda_2) E^{-1}\left[\vartheta^{\gamma} E\left[|\zeta - \xi|\right]\right] \right\|, \\ &= (\lambda_1 + \lambda_2)\vartheta^{\gamma+2} \|\zeta - \xi\|. \end{aligned} \quad (20)$$

f is a contraction as $0 < (\lambda_1 + \lambda_2)\vartheta^{\gamma+2} \le 1$, from the Banach fixed point theorem.

3.3 Convergence Theorem

Theorem 3.2: Let $\zeta_n(\tau)$ and $\zeta_m(\tau)$ be in Banach space B. If there exists a positive constant $\delta = (\lambda_1 + \lambda_2)\vartheta^{\gamma+2} \in (0, 1)$ such that $\|\zeta_{n+1}(\tau)\| \le \delta \|\zeta_n(\tau)\|$ for all $\tau \in \Omega$ with $\|\zeta_1(\tau) - \zeta_0(\tau)\| \le \infty,$ then the sequence defined by Eq. (17) with $\zeta_0(\tau)=\zeta(0)$ converges to $\zeta(\tau)$, i.e the exact solution of Eq. (7).

Proof: To prove this theorem, it suffices to show that $\zeta_n(\tau)$ is the Cauchy sequence in Banach space B.

$$\begin{aligned}\|\zeta_n(\tau)-\zeta_m(\tau)\| &= \max_{\tau\in\Omega}\left|\zeta_n(\tau)-\zeta_m(\tau)\right|,\\ &\le \max_{\tau\in\Omega}\left|E^{-1}\left[\vartheta^{\gamma}E\left[L\zeta_n(\tau)+N\zeta_n(\tau)\right]\right]-E^{-1}\left[\vartheta^{\gamma}E\left[L\zeta_m(\tau)+N\zeta_m(\tau)\right]\right]\right|,\\ &\le \max_{\tau\in\Omega}\left|E^{-1}\left[\vartheta^{\gamma}E\left[L\zeta_n(\tau)-L\zeta_m(\tau)\right]\right]+E^{-1}\left[\vartheta^{\gamma}E\left[L\zeta_n(\tau)-N\zeta_m(\tau)\right]\right]\right|,\\ &\le \max_{\tau\in\Omega}\left|\left(\lambda_1+\lambda_2\right)E^{-1}\left[\vartheta^{\gamma}E\left[\left|\zeta_n-\zeta_m\right|\right]\right]\right|,\\ &= \left(\lambda_1+\lambda_2\right)\vartheta^{\gamma+2}\left\|\zeta_n-\zeta_m\right\|.\end{aligned} \tag{21}$$

Let $n=m+1$, then

$$\left\|\zeta_{m+1}(\tau)-\zeta_m(\tau)\right\|\le\delta\left\|\zeta_m(\tau)-\zeta_{m-1}(\tau)\right\|\le\delta^2\left\|\zeta_{m-1}(\tau)-\zeta_{m-2}(\tau)\right\|\le\cdots\le\delta^m\left\|\zeta_1(\tau)-\zeta_0(\tau)\right\|, \tag{22}$$

$$\delta=\left(\lambda_1+\lambda_2\right)\vartheta^{\gamma+2}. \tag{23}$$

From the triangle inequality, we have

$$\begin{aligned}\left\|\zeta_n(\tau)-\zeta_m(\tau)\right\| &= \left\|\zeta_{m+1}(\tau)-\zeta_m(\tau)+\zeta_{m+2}(\tau)-\zeta_{m+1}(\tau)+\cdots+\zeta_n(\tau)-\zeta_{n-1}(\tau)\right\|\\ &\le \left\|\zeta_{m+1}(\tau)-\zeta_m(\tau)\right\|+\left\|\zeta_{m+2}(\tau)-\zeta_{m+1}(\tau)\right\|+\cdots+\left\|\zeta_n(\tau)-\zeta_{n-1}(\tau)\right\|\\ &\le \delta^m\left\|\zeta_1(\tau)-\zeta_0(\tau)\right\|+\delta^{m+1}\left\|\zeta_1(\tau)-\zeta_0(\tau)\right\|+\cdots+\delta^{n-1}\left\|\zeta_1(\tau)-\zeta_0(\tau)\right\|\\ &= \delta^m\left(1+\delta+\cdots+\delta^{n-m-1}\right)\left\|\zeta_1(\tau)-\zeta_0(\tau)\right\|\\ &\le \delta^m\left(\frac{1-\delta^{n-m}}{1-\delta}\right)\left\|\zeta_1(\tau)-\zeta_0(\tau)\right\|.\end{aligned} \tag{24}$$

Since $0<\delta<1$, so $1-\delta^{n-m}<1$, then

$$\left\|\zeta_n(\tau)-\zeta_m(\tau)\right\|=\delta^m\left(\frac{1-\delta^{n-m}}{1-\delta}\right)\left\|\zeta_1(\tau)-\zeta_0(\tau)\right\|. \tag{25}$$

But $\left\|\zeta_1(\tau)-\zeta_0(\tau)\right\|<\infty$, then $\left\|\zeta_n(\tau)-\zeta_m(\tau)\right\|\to 0$ as $n\to\infty$. We conclude that $\left\{\zeta_n(\tau)\right\}$ is a Cauchy sequence in the Banach space B. Therefore the sequence converges.

4. Applications

This study considers the logistic differential equation with a Caputo fractional derivative of order γ and the proportional time delay. Consider

$${}_0^C D_\tau^\gamma\zeta(\tau)=\varepsilon\zeta(\tau)\left(1-\zeta(\rho\tau)\right),\quad \rho\ge 0, \tag{26}$$

With the initial condition

$$\zeta_0(\tau)=\kappa. \tag{27}$$

$\rho\in[0,1]$, which is the initial population. When $\rho>0$ or $\rho\neq 1$, a pantograph-type equation is obtained in (26), differential equations with variable delay $t=t(\tau)$ is obtained, where $t(\tau)=(1-\rho)\tau$ with $t(\tau)>0$. The special case of delay differential equations with changing delay for $0<\rho<1$ is Equation (26).

Case 4.1. Consider $\rho=0$, Equation (26) gives

$${}_0^C D_\tau^\gamma\zeta(\tau)=\varepsilon(1-\kappa)\zeta(\tau),\quad \zeta_0(\tau)=\kappa. \tag{28}$$

The exact solution of Eq. (28) is given:

$$\zeta(\tau)=\kappa E\left(\varepsilon(1-\kappa)\tau^{\gamma}\right). \tag{29}$$

To solve the linear fractional ordinary differential equation (28) using the homotopy perturbation Elzaki transform method, apply the recursive formula from equation (15).

$$\begin{aligned}&\zeta_0(\tau)=\kappa,\\&\zeta_{m+1}(\tau)=\varepsilon\,(1-\kappa)E^{-1}\left[\vartheta^{\gamma}E\left[\zeta_m(\tau)\right]\right],\, m=0,1,2,\dots\,.\end{aligned} \tag{30}$$

If the recursive solution in equation (30) is described, then based on equation (5) and the inverse of Elzaki transform, the following solution is obtained:

$$\begin{aligned}&\zeta_0(\tau)=\kappa,\\&\zeta_1(\tau)=\varepsilon\kappa(1-\kappa)E^{-1}\left[\vartheta^{\gamma}E[1]\right]=\varepsilon\kappa(1-\kappa)E^{-1}\left[\vartheta^{\gamma+2}\right]=\varepsilon\kappa(1-\kappa)\frac{\tau^{\gamma}}{\Gamma(\gamma+1)},\\&\zeta_2(\tau)=\varepsilon^2\kappa(1-\kappa)^2E^{-1}\left[\vartheta^{\gamma}E\left[\frac{\tau^{\gamma}}{\Gamma(\gamma+1)}\right]\right]=\varepsilon^2\kappa(1-\kappa)^2E^{-1}\left[\vartheta^{2\gamma+2}\right]=\varepsilon^2\kappa(1-\kappa)^2\frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)},\\&\zeta_3(\tau)=\varepsilon^3\kappa(1-\kappa)^3\frac{\tau^{3\gamma}}{\Gamma(3\gamma+1)},\\&\zeta_4(\tau)=\varepsilon^4\kappa(1-\kappa)^4\frac{\tau^{4\gamma}}{\Gamma(4\gamma+1)}.\end{aligned} \tag{31}$$

Thus, the following approach solutions are obtained:

$$\begin{aligned}\zeta(\tau)&=\zeta_0(\tau)+\zeta_1(\tau)+\zeta_2(\tau)+\cdots\\&=\kappa+\varepsilon\kappa(1-\kappa)\frac{\tau^{\gamma}}{\Gamma(\gamma+1)}+\varepsilon^2\kappa(1-\kappa)^2\frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)}+\varepsilon^3\kappa(1-\kappa)^3\frac{\tau^{3\gamma}}{\Gamma(3\gamma+1)}+\varepsilon^4\kappa(1-\kappa)^4\frac{\tau^{4\gamma}}{\Gamma(4\gamma+1)}+\cdots.\end{aligned} \tag{32}$$

Case 4.2. Consider $\rho=1$, Equation (26) gives

$$ {}_0^CD_\tau^{\gamma}\zeta(\tau)=\varepsilon\zeta(\tau)(1-\zeta(\tau)),\ \zeta_0(\tau)=\kappa. \tag{33}$$

The exact solution of Eq. (33) is given:

$$\zeta(\tau)=\frac{\kappa}{(1-\kappa)\left(e^{-\varepsilon\kappa}\right)+\kappa}. \tag{34}$$

To solve the linear fractional ordinary differential equation (33) using the homotopy perturbation Elzaki transform method, apply the recursive formula from equation (15).

$$\begin{aligned}&\zeta_0(\tau)=\kappa,\\&\zeta_{m+1}(\tau)=\varepsilon E^{-1}\left[\vartheta^{\gamma}E\left[\zeta_m(\tau)-B_m\right]\right],\, m=0,1,2,\dots.\end{aligned} \tag{35}$$

The nonlinear term is represented by

$$N(\zeta)=\zeta^2 \tag{36}$$

Using equation (36) in equation (14), we obtain

$$B_0=\zeta_0^2(\tau),\quad B_1=2\zeta_0(\tau)\zeta_1(\tau),\quad B_2=2\zeta_0(\tau)\zeta_2(\tau)+\zeta_1^2(\tau),\dots\,. \tag{37}$$

If the recursive solution in equation (35) is described, then based on equation (5), equation (36) and the inverse of Elzaki transform, the following solution is obtained:

$$\zeta_0(\tau)=\kappa,$$
$$\zeta_1(\tau)=\varepsilon E^{-1}\left[\vartheta^{\gamma}E\left[\zeta_0(\tau)-B_0\right]\right]=\varepsilon E^{-1}\left[\vartheta^{\gamma}E\left[\zeta_0(\tau)-\zeta_0^2(\tau)\right]\right]=\varepsilon\kappa(1-\kappa)E^{-1}\left[\vartheta^{\gamma+2}\right]=\frac{\varepsilon\kappa(1-\kappa)\tau^{\gamma}}{\Gamma(\gamma+1)},$$
$$\zeta_2(\tau)=\varepsilon^2E^{-1}\left[\vartheta^{\gamma}E\left[\zeta_1(\tau)-B_1\right]\right]=\varepsilon^2E^{-1}\left[\vartheta^{\gamma}E\left[\zeta_1(\tau)-2\zeta_0(\tau)\zeta_1(\tau)\right]\right]=\varepsilon^2\kappa(1-\kappa)(1-2\kappa)E^{-1}\left[\vartheta^{2\gamma+2}\right]$$
$$=\frac{\varepsilon^2\kappa(1-\kappa)(1-2\kappa)\tau^{2\gamma}}{\Gamma(2\gamma+1)},$$

$$\zeta_3(\tau)=\frac{\varepsilon^3\kappa(1-\kappa)\left((1-2\kappa)^2\Gamma^2(\gamma+1)-\kappa(1-\kappa)\Gamma(2\gamma+1)\right)\tau^{3\gamma}}{\Gamma^2(\gamma+1)\Gamma(3\gamma+1)},$$
$$\zeta_4(\tau)=\frac{\varepsilon^4\kappa^2(1-\kappa)^2(1-2\kappa)\left(\Gamma(2\gamma+1)-2\Gamma(3\gamma+1)\Gamma(\gamma+1)\right)\tau^{4\gamma}}{\Gamma^2(\gamma+1)\Gamma(2\gamma+1)\Gamma(4\gamma+1)}+\frac{\varepsilon^4\kappa(1-\kappa)(1-2\kappa)^3\Gamma^2(\gamma+1)\tau^{4\gamma}}{\Gamma^2(\gamma+1)\Gamma(4\gamma+1)}. \tag{38}$$

Thus, the following approach solutions are obtained:

$$\begin{aligned}\zeta(\tau)&=\zeta_0(\tau)+\zeta_1(\tau)+\zeta_2(\tau)+\cdots\\&=\kappa+\frac{\varepsilon\kappa(1-\kappa)\tau^{\gamma}}{\Gamma(\gamma+1)}+\frac{\varepsilon^2\kappa(1-\kappa)(1-2\kappa)\tau^{2\gamma}}{\Gamma(2\gamma+1)}+\frac{\varepsilon^3\kappa(1-\kappa)\left((1-2\kappa)^2\Gamma^2(\gamma+1)-\kappa(1-\kappa)\Gamma(2\gamma+1)\right)\tau^{3\gamma}}{\Gamma^2(\gamma+1)\Gamma(3\gamma+1)}\\&+\frac{\varepsilon^4\kappa^2(1-\kappa)^2(1-2\kappa)\left(\Gamma(2\gamma+1)-2\Gamma(3\gamma+1)\Gamma(\gamma+1)\right)\tau^{4\gamma}}{\Gamma^2(\gamma+1)\Gamma(2\gamma+1)\Gamma(4\gamma+1)}+\frac{\varepsilon^4\kappa(1-\kappa)(1-2\kappa)^3\Gamma^2(\gamma+1)\tau^{4\gamma}}{\Gamma^2(\gamma+1)\Gamma(4\gamma+1)}+\cdots.\end{aligned} \tag{39}$$

Case 4.3. Consider $0<\rho<1$, Equation (26) gives

$${}_0^CD_\tau^{\gamma}\zeta(\tau)=\varepsilon\zeta(\tau)\left(1-\zeta(\rho\tau)\right),\ \zeta_0(\tau)=\kappa. \tag{40}$$

To solve the linear fractional ordinary differential equation (40) using the homotopy perturbation Elzaki transform method, apply the recursive formula from equation (15).

$$\begin{aligned}&\zeta_0(\tau)=\kappa,\\&\zeta_{m+1}(\tau)=\varepsilon E^{-1}\left[\vartheta^{\gamma}E\left[\zeta_m(\tau)-B_m\right]\right],m=0,\ 1,\ 2,\ \ldots.\end{aligned} \tag{41}$$

The nonlinear term be represented by

$$N(\zeta)=\zeta^2(\rho\tau) \tag{42}$$

Using equation (42) in equation (14), we obtain

$$B_0=\zeta_0^2(\tau),\quad B_1=2\zeta_0(\tau)\zeta_1(\rho\tau),\quad B_2=2\zeta_0(\tau)\zeta_2(\rho\tau)+\zeta_1^2(\rho\tau),\ldots. \tag{43}$$

If the recursive solution in equation (41) is described, then based on equation (5), equation (42) and the inverse of the Elzaki transform, the following solution is obtained:

$$\zeta_0(\tau)=\kappa,$$
$$\zeta_1(\tau)=\varepsilon E^{-1}\left[\vartheta^{\gamma}E\left[\zeta_0(\tau)-B_0\right]\right]=\varepsilon E^{-1}\left[\vartheta^{\gamma}E\left[\zeta_0(\tau)-\zeta_0^2(\tau)\right]\right]=\varepsilon\,\kappa(1-\kappa)E^{-1}\left[\vartheta^{\gamma+2}\right]=\frac{\varepsilon\,\kappa\,(1-\kappa)\tau^{\gamma}}{\Gamma(\gamma+1)},$$
$$put\ \zeta_1(\rho\tau)=\frac{\varepsilon\,\kappa\,\rho^{\gamma}(1-\kappa)\tau^{\gamma}}{\Gamma(\gamma+1)},$$
$$\zeta_2(\tau)=\varepsilon^2E^{-1}\left[\vartheta^{\gamma}E\left[\zeta_1(\tau)-B_1\right]\right]=\varepsilon^2E^{-1}\left[\vartheta^{\gamma}E\left[\zeta_1(\tau)-2\zeta_0(\tau)\,\zeta_1(\rho\tau)\right]\right]$$
$$=\varepsilon^2\,\kappa\,(1-\kappa)\,(1-2\kappa)\,E^{-1}\left[\vartheta^{2\gamma+2}\right]=\frac{\varepsilon^2\,\kappa\,(1-\kappa)\left(1-2\,\kappa\,\rho^{\gamma}\right)\tau^{2\gamma}}{\Gamma(2\gamma+1)},$$
$$put\ \zeta_2(\rho\tau)=\frac{\varepsilon^2\,\kappa\,\rho^{2\gamma}(1-\kappa)\left(1-2\kappa\,\rho^{\gamma}\right)\tau^{2\gamma}}{\Gamma(2\gamma+1)},$$
$$\zeta_3(\tau)=\frac{\varepsilon^3\,\kappa\,(1-\kappa)\left(\left(1-2\kappa\,\rho^{\gamma}\right)^2\Gamma^2(\gamma+1)-\kappa(1-\kappa)\,\rho^{2\gamma}\,\Gamma(2\gamma+1)\right)\tau^{3\gamma}}{\Gamma^2(\gamma+1)\,\Gamma(3\gamma+1)},$$
$$\zeta_4(\tau)=\frac{\varepsilon^4\,\kappa^2\,(1-\kappa)^2\left(1-2\kappa\,\rho^{3\gamma}\right)\rho^{2\gamma}\left(\Gamma(2\gamma+1)-2\rho^{3\gamma}\,\Gamma(3\gamma+1)\Gamma(\gamma+1)\right)\tau^{4\gamma}}{\Gamma^2(\gamma+1)\,\Gamma(2\gamma+1)\,\Gamma(4\gamma+1)}$$
$$+\frac{\varepsilon^4\,\kappa\,(1-\kappa)\left(1-2\,\kappa\,\rho^{\gamma}\right)^3\Gamma^2(\gamma+1)\,\tau^{4\gamma}}{\Gamma^2(\gamma+1)\,\Gamma(4\gamma+1)}. \tag{44}$$

Thus, the following approach solutions are obtained:

$$\zeta(\tau)=\zeta_0(\tau)+\zeta_1(\tau)+\zeta_2(\tau)+\cdots$$
$$=\kappa+\frac{\varepsilon\,\kappa(1-\kappa)\,\tau^{\gamma}}{\Gamma(\gamma+1)}+\frac{\varepsilon^2\kappa\,(1-\kappa)\left(1-2\,\kappa\,\rho^{\gamma}\right)\tau^{2\gamma}}{\Gamma(2\gamma+1)}+\frac{\varepsilon^3\,\kappa\,(1-\kappa)\left(\left(1-2\,\kappa\,\rho^{\gamma}\right)^2\Gamma^2(\gamma+1)-\kappa(1-\kappa)\,\rho^{2\gamma}\,\Gamma(2\gamma+1)\right)\tau^{3\gamma}}{\Gamma^2(\gamma+1)\,\Gamma(3\gamma+1)} \tag{45}$$
$$+\frac{\varepsilon^4\,\kappa^2(1-\kappa)^2\left(1-2\,\kappa\,\rho^{3\gamma}\right)\rho^{2\gamma}\left(\Gamma(2\gamma+1)-2\rho^{3\gamma}\,\Gamma(3\gamma+1)\Gamma(\gamma+1)\right)\tau^{4\gamma}}{\Gamma^2(\gamma+1)\,\Gamma(2\gamma+1)\,\Gamma(4\gamma+1)}+\frac{\varepsilon^4\,\kappa\,(1-\kappa)\left(1-2\,\kappa\,\rho^{\gamma}\right)^3\Gamma^2(\gamma+1)\,\tau^{4\gamma}}{\Gamma^2(\gamma+1)\,\Gamma(4\gamma+1)}+\cdots.$$

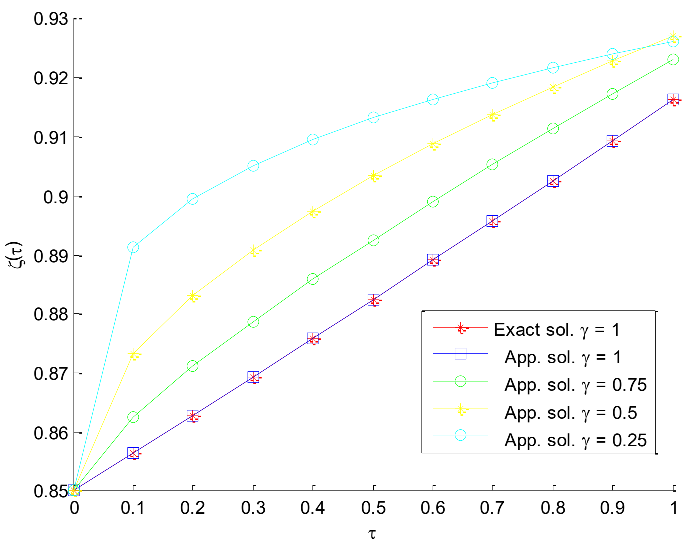
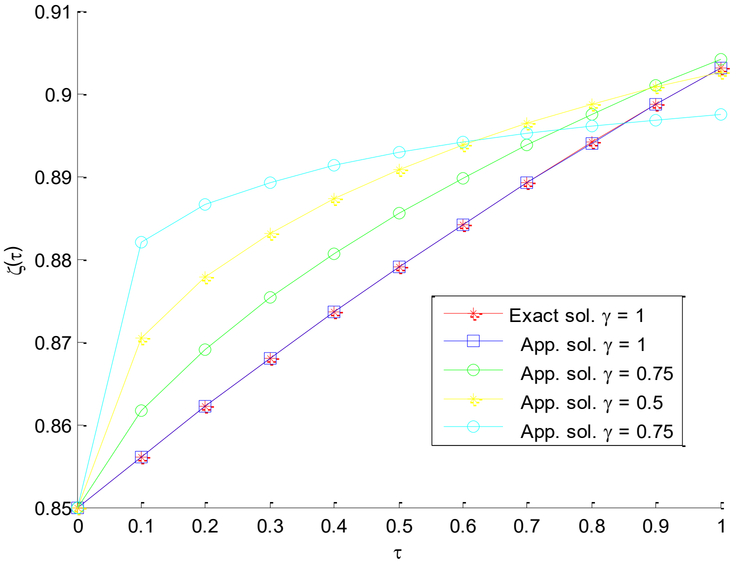


Figure 1. 2D-Surface behavior of the 5th HPETM and the exact solution for Eq. (32), when $\tau=[0,1]$, $\zeta_0(\tau)=0.85$ and $\varepsilon=0.5$ for diverse γ.



Figue 2. 2D-Surface behavior of the 5th HPETM and the exact solution for Eq. (39), when $\tau = [0,1]$, $\zeta_0(\tau) = 0.85$ and $\varepsilon = 0.5$ for diverse γ.

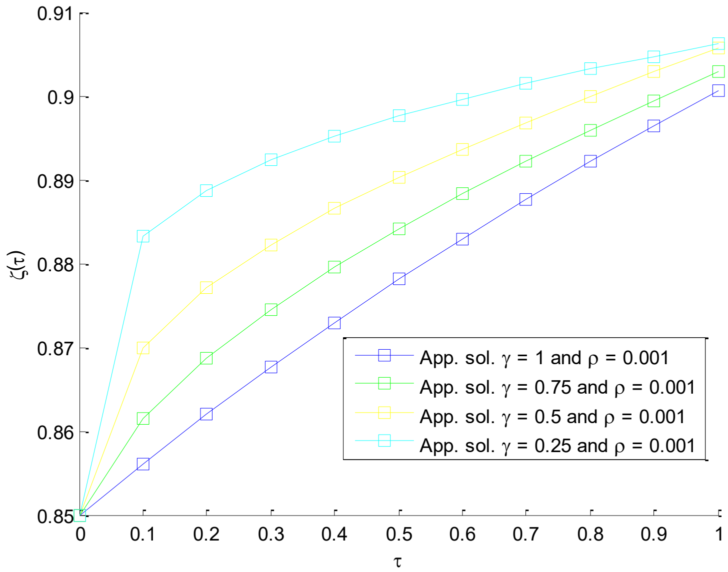


Figure 3. 2D-Surface behavior of the 5th HPETM for the logistic model with fractional derivative and time delay for Eq. (45), when $\tau = [0,1]$, $\zeta_0(\tau) = 0.85$ and $\varepsilon = 0.5$ for diverse γ.

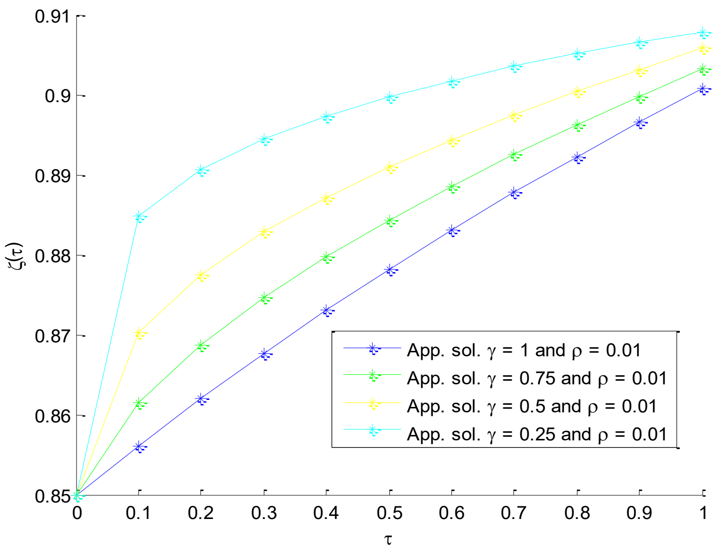


Figure 4. 2D-Surface behavior of the 5th HPETM for the logistic model with fractional derivative and time delay for Eq. (45), when $\tau=[0,1]$, $\zeta_0(\tau)=0.85$ and $\varepsilon=0.5$ for diverse γ.

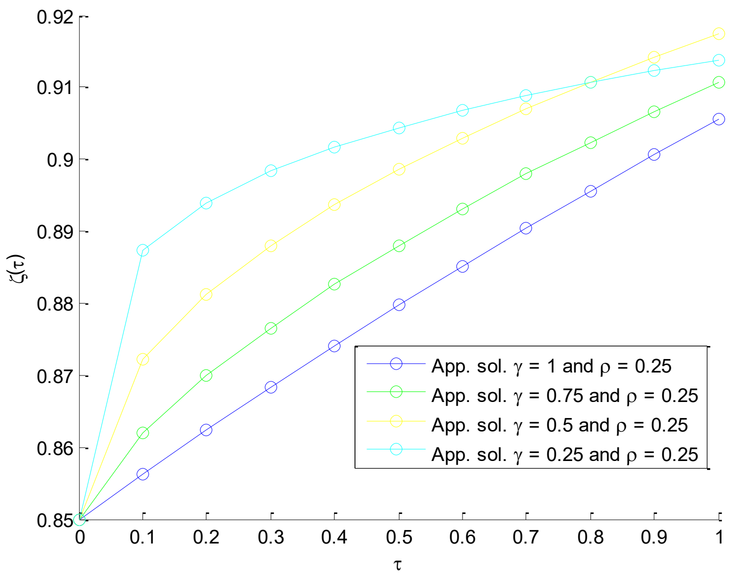


Figure 5. 2D-Surface behavior of the 5th HPETM for the logistic model with fractional derivative and time delay for Eq. (45), when $\tau=[0,1]$, $\zeta_0(\tau)=0.85$ and $\varepsilon=0.5$ for diverse γ.

5. Numerical and Graphical Discussions

In this section, we demonstrate the agreement between the approximate and exact solutions obtained using the Homotopy Perturbation Elzaki Transform Method (HPETM). This agreement is evident in the graph discussions. Selecting the parameters $\varepsilon=0.5$, $\kappa=0.85$, and using MATLAB 2013. Fig. 1 illustrates the exact and approximate solutions for the fractional logistic differential equation given in Eqs. (29), (32), which solve Eq. (28) for different values of $\gamma=0.25, 0.5, 0.75, 1,$ and the parameter $\rho=0.$ Fig. 2 illustrates the exact and approximate solutions for the fractional logistic differential equation given in Eqs. (34), (39), which solve Eq. (33), for different values of $\gamma=0.25, 0.5, 0.75, 1,$ and the parameter $\rho=1.$ Fig. 3, 4 and 5 illustrate the approximate solutions for the fractional logistic differential equation and time delay $\rho\in(0,1)$ given in Eq. (45), which solves Eq.

(49) respectively, for different values of $\gamma = 0.25, 0.5, 0.75, 1,$ and the parameter $\rho = 0.001, 0.01, \ 0.25.$ The findings obtained in Eqs. (32), (39), and (45) are comparable to those produced by Sumudu decomposition method (SDM), Adams-type predictor–corrector method (ATPCM), variational iteration method (VIM) and the finite difference method (FDM). Finally, the findings show that the homotopy perturbation Elzaki transform method (HPETM) is a superior development of previous numerical approaches.

6. Conclusion

In this paper, the homotopy perturbation Elzaki transform method (HPETM) is effectively and convergently applied for the logistic differential equation with the Caputo-fractional derivative and time delay $\rho \in (0,1)$. The outcomes are in good comparison to those of SDM, ATPCM, VIM and FDM. Finally, the findings show that the HPETM is a superior development of previous numerical approaches.

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