Derna University Journal of Applied Sciences
Vol. 2, Issue 2, 2025
ISSN: 3079-0719 (Online) | DOI: https://doi.org/10.58987/8nzmrz73

A More General Existence Theorem for Weak Solutions in Banach Spaces

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Received: May 21, 2025; **Accepted**: Dec 13, 2025

Abstract—This paper contains a Kubiaczyks fixed point theorem for weakly continuous solutions of a Volterra-Stieltjes integral equation in nonreflexive Banach spaces. The Kubiaczyk's fixed point theorem is applied to establish the existence of a solution. A compactness type condition in connection with the weak topology is used. The existence is proved by defining an integral operator F which is shown to be weakly sequentially continuous on a closed, convex, and equicontinuous subset Q of weakly continuous functions C_ω. The measure of weak noncompactness β is a crucial tool in this proof. As an application, the existence of a weakly continuous solution for Cauchy problems in Banach spaces is deduced. It is also shown that a fractional integral equation in the sense of weakly Riemann is a special case of the integral equation of Volterra -Stieltjes type. This leads to the existence of weak solutions in C_ω for Cauchy problems involving the fractional Caputo weak derivative.

Keywords—Cauchy problem; Volterra-Stieltjes integral; weak continuity; Measure of weak noncompactness

1. Introduction and Preliminaries

Let E be a nonreflexive Banach space with norm $\|\cdot\|$ with its dual E^*, and we will denote by $E_\omega = (E, \omega) = \big(E, \sigma(E, E^*)\big)$ the space E with its weak topology. Denote by $C[I, E_\omega]$ the Banach space of weakly continuous functions from $I = [0, a]$ to E_ω endowed with the topology of weak uniform convergence.

Consider the integral equation of Volterra-Stieltjes type

$$x(t) = p(t) + \int_0^t f\big(s, x(s)\big)\, d_s g(t, s), t \in I. \tag{1}$$

The integral equations of Volterra-Stieltjes type in Banach spaces have been developed extensively. Such equations serve as fundamental models for various phenomena in applied mathematics and physics, often linking existence results to the analysis of differential equations and their fractional counterparts. A key application derived from the study of this integral equation is the existence of a weakly continuous solution for Cauchy problems in Banach spaces.

The integral equations of Volterra-Stieltjes type in Banach spaces have been developed extensively. However almost all of the work was done using the strong topology (see, for example, [1–8]) while the study of our problem involving the weak topology is lagging behind.

The results of Szep [9] and Salem and El-Sayed [10] were extended to nonreflexive Banach spaces by Boundourides [11], Cramer *et al.* [12] and by Agarwal *et al.* [13,14].

Recently, in [15] and [16] the authors studied the existence of weak solutions in reflexive Banach spaces for the Volterra-Stieltjes integral equation (1), where f is assumed to be weakly-weakly continuous.

Here we are concerned with the Volterra-Stieltjes integral equation (1). The existence of weak solutions in C_ω of the integral equation (1) in the nonreflexive Banach space E is proved. As applications of our result, we show that a fractional integral equation in the sense of weakly Riemann is a special case of the integral equation of Volterra-Stieltjes type (1).

Moreover, in this study we deduce the existence of weak solutions in C_ω for the Cauchy problems

$$D_\omega^\alpha x(t) = f\big(t, x(t)\big),\ x(0) = x_0,\ t \in (0, a] \tag{2}$$

where $D_\omega^\alpha x(.)$ is the fractional Caputo weak derivative of the function $x : I \to E$.

$$\frac{dx}{dt} = f(t, x(t)),\ x(0) = x_0,\ t \in (0, a] \tag{3}$$

Now, let $r > 0$ be given and define the set

$$B_r = \{x(t) \in E;\ \| x\|_0 \leq r\}.$$

Lemma 1: Let $f : I \times B_r \to E$ be weakly-weakly continuous, then

- For each $t \in I$; $f(t, .)$ is weakly continuous, hence weakly sequentially continuous (see [17]),
- For each weakly continuous $x : I \to B_r$; $f(., x(.))$ is weakly continuous on I (see [18]),
- f is norm bounded, i.e. there exists an M_r such that $\|f(t,x)\| \leq M_r$ for all $(t, x) \in I \times B_r$ (see [9]).

  Assume that a nondecreasing function which is continuous $h : R^+ \to R^+$ satisfies the following conditions

  $A_1)$ $h(0) = 0$

  $A_2)$ $z(t) = 0$ is the unique continuous solution of the integral inequality

$$u(t) \leq \int_0^t h\big(u(s)\big)\, d_s g(t,s), t \in I$$

Satisfying the condition

$$u(0) = 0$$

Proposition 1: Let E be a normed space with $y \in E, y \neq 0$. Then there exists a $\varphi \in E^*$ with $\|\varphi\| = 1$, $\|y\| = \varphi(y)$.

Further on, denote by m_E the family of nonempty and bounded subsets of E.

In this paper by a measure of weak noncompactness, we will understand a function β.

1. $\beta(A) = 0 \Leftrightarrow A$ is relatively weakly compact in E,
2. $A \subset B \Rightarrow \beta(A) \leq \beta(B)$,
3. $\beta(A \cup \{x\}) = \beta(A), x \in E$,
4. $\beta(\lambda A) = |\lambda|.\beta(A), \lambda \in R$,
5. $\beta(A + B) \leq \beta(A) + \beta(B)$,
6. $\beta(A \cup B) \leq \max(\beta(A), \beta(B))$.

It is necessary to remark that if β has these properties, then the following lemma is true.

Lemma 2 ([19,20]): Let $V \subset C_\omega$ be a family of strongly equicontinuous functions. Then the function $t \mapsto v(t) = \beta(V(t))$ is continuous and $\beta(V(I)) = \sup\{\beta(V(t)): t \in I\}$.

Now we have the following Kubiaczyk's fixed point theorem that will be needed in this paper (see [21])

Theorem 1: Let Q be a closed convex and equicontinuous subset of a metrizable locally convex vector space E and let F be a weakly sequentially continuous mapping of Q into itself. Suppose that for some $x(\cdot) \in Q$, the following implication holds for every subset $V \subseteq Q$:

$$\bar{V} = \overline{conv}(F(V) \cup \{x\}) \Rightarrow V \quad (4)$$

and that these sets are relatively weakly compact. Then Fhas a fixed point.

2. Volterra-Stieltjes integral equation

Let us denote by $J = [0, T]$ where $T = \min\{a, \frac{r}{M_r}\}$ and $C_\omega = C[J, E_\omega]$. We begin this section by using Theorem 1 to establish existence of weak solutions $x(\cdot) \in C_\omega m$for the Volterra-Stieltjes integral equation (1) will be sought in the nonreflexive Banach space E.

By a weak solution to (1) we mean the weakly continuous function x which satisfies the integral equation (1). This is equivalent to finding $x(\cdot) \in C_\omega$with

$$\varphi(x(t)) = \varphi\left(p(t) + \int_0^t f(s, x(s)) d_s g(t, s)\right), t \in I.$$

Denoted Λ by

$$\Lambda = \{(t, s): 0 \leq s \leq t \leq T\}.$$

To facilitate our discussion, let us first state the following assumptions:

i. $p: I \to E$ is weakly continuous.
ii. The function $f: I \times B_r \to E$ is weakly-weakly continuous.
iii. The function $g: \Lambda \to R$ is continuous on Λ.
iv. The function $s \to g(t, s)$ is of bounded variation on $[0, t]$ for each fixed $t \in J$.
v. For any $\epsilon > 0$ there exists $\delta > 0$ for all $t_1, t_2 \in J$ such that $t_1 < t_2$ and $t_2 - t_1 \leq \delta$ the following inequality holds
$\bigvee_0^{t_1}[g(t_2, s) - g(t_1, s)] \leq \epsilon$.

vi. $g(t, 0) = 0$ for any $t \in J$.

Obviously, we will assume that g satisfies assumptions (iii)-(v). For our purposes we will only need the following lemmas.

Lemma 3 [7]: The function $z \to \bigvee_{s=0}^{z} g(t,s)$ is continuous on $[0,t]$ for any fixed $t \in J$.

Lemma 4 [7]: For an arbitrary fixed $0 < t_2 \in J$ and for any $\epsilon > 0$, there exists $\delta > 0$ such that if

$$t_1 \in I, t_1 < t_2, t_2 - t_1 \leq \delta$$

then

$$\bigvee_{s=t_1}^{t_2} g(t_2, s) \leq \epsilon.$$

Lemma 5 [7]: The function $t \to \bigvee_{s=0}^{t} g(t,s)$ is continuous on J. Then there exists a finite positive constant K such that

$$K = \sup \{\bigvee_{s=0}^{t} g(t,s) : t \in J\}.$$

First, we shall show the following main theorem,

Theorem 2: Under the assumptions (i)-(vi), if

$$\beta(f(J \times X)) \leq h(\beta(X))$$

for each $X \subset B_r$, $J \subset I$, then there exists at least one weak solution $x(\cdot) \in C_\omega$.

Proof: We define the integral operator $F: C_\omega \to C_\omega$associated to the integral equation (1) by

$$Fx(t) = p(t) + \int_0^t f\big(s, x(s)\big) d_s g(t,s), t \in J \tag{5}$$

According to Lemma 1 for every weakly continuous function $x(.), f(., x(.))$ is weakly continuous, means that $\varphi(f(., x(.)))$ is continuous, for any $\varphi \in E^*, g$ is of bounded variation. Hence $f(., x(.))$ is weakly Riemann-Stieltjes integrable on $J$with respect to $s \to g(t,s)$. Thus F makes sense.

Now, define the closed, convex, bounded, equicontinuous set Q of all weakly continuous functions $x(\cdot): J \to B_r$ such that

$$\|x(t_2) - x(t_1)\| \leq \|p(t_2) - p(t_1)\| + MN\epsilon + M \bigvee_{s=t_1}^{t_2} g(t_2, s)$$

For notational purposes$\|x\|_0 = sup_{t \in J} \|x(t)\|$.

We claim that $F: Q \to Q$ is weakly sequentially continuous and FQ is weakly relatively compact. Once the claim is established, then Theorem 1 with C_ω guarantees a fixed point of F, and hence (1) has a solution in C_ω.

First, we show that FQ is an equicontinuous. Let $t_1,\ t_2 \in I,\ t_2 > t_1,\ x(.) \in Q$, without loss of generality, assume $Fx(t_2) - Fx(t_1) \neq 0$

$$\|Fx(t_2) - Fx(t_1)\| = \varphi(Fx(t_2) - Fx(t_1))$$
$$\leq |\varphi(p(t_2) - p(t_1)| + \left|\int_0^{t_2} \varphi(f(s,x(s)))d_s g(t_2,s) - \int_0^{t_1} \varphi(f(s,x(s)))d_s g(t_1,s)\right|$$

$$\leq \|p(t_2) - p(t_1)\| + \left|\int_0^{t_1} \varphi(f(s,x(s)))d_s g(t_2,s) + \int_{t_1}^{t_2} \varphi(f(s,x(s)))d_s g(t_2,s) - \int_0^{t_1} \varphi(f(s,x(s)))d_s g(t_1,s)\right|$$

$$\leq \|p(t_2) - p(t_1)\| + \left|\int_0^{t_1} \varphi(f(s,x(s)))\, d_s[g(t_2,s) - g(t_1,s)\,]\right| + \left|\int_{t_1}^{t_2} \varphi(f(s,x(s)))d_s g(t_2,s)\right|$$

$$\leq \|p(t_2) - p(t_1)\| + \int_0^{t_1} \left|\varphi\left(f(s,x(s))\right)\right|\ d_s\,[\bigvee_{z=0}^{s}(g(t_2,z) - g(t_1,z))\,] + \int_{t_1}^{t_2} \left|\varphi\left(f(s,x(s))\right)\right|\ d_s[\bigvee_{z=0}^{s} g(t_2,z)]$$

$$\leq \|p(t_2) - p(t_1)\| + M_r \int_0^{t_1} d_s\,[\bigvee_{z=0}^{s}(g(t_2,z) - g(t_1,z))]\ \ + M_r \int_{t_1}^{t_2} d_s[\bigvee_{z=0}^{s} g(t_2,z)]$$

$$\leq \|p(t_2) - p(t_1)\| + M_r \bigvee_{s=0}^{t_1}(g(t_2,s) - g(t_1,s)) + M_r[\bigvee_{s=0}^{t_2} g(t_2,s) - \bigvee_{s=0}^{t_1} g(t_2,s)]$$

$$\leq \|p(t_2) - p(t_1)\| + M_r N(\epsilon) + M_r \bigvee_{s=t_1}^{t_2} g(t_2,s)$$

Where

$$N(\epsilon) = \sup\{\bigvee_{s=0}^{t_1}\big(g(t_2,s) - g(t_1,s)\big):\ t_1,t_2 \in\ I, t_2 >\ t_1, t_2 -\ t_1 \leq \epsilon\}$$

Hence

$$\|Fx(t_2) - Fx(t_1)\| \leq \|p(t_2) - p(t_1)\| + M_r N(\epsilon) + M_r \vee_{s=t_1}^{t_2} g(t_2,s) \qquad (6)$$

Thus $Fx(\cdot) \in C_\omega$ and the operator F is well defined. Let $t \in J$. By the Hahn-Banach theorem, there exists $\varphi \in E^*$ such that $\|\varphi\| = 1$, and $\|\mathrm{F}x(\mathrm{t})\| = \varphi(Fx(t))$. Then, using the assumptions $(i) - (vi)$, we have

$$\|Fx(t)\| = \varphi\big(Fx(t)\big) \leq \big|\varphi\big(p(t)\big)\big| + \left|\varphi(\int_0^t f\big(s,x(s)\big)d_s g(t,s))\right|$$

$$\leq \|p(t)\| + \int_0^t \left|\varphi\left(f\big(s,x(s)\big)\right)\right|\ d_s\,(\bigvee_{z=0}^{s} g(t,z))$$

$$\leq \|p(t)\| + M_r \int_0^t d_s\,(\bigvee_{z=0}^{s} g(t,z)) \leq \|p(t)\| + M_r \bigvee_{s=0}^{t} g(t,s)$$

$$\leq \|p\|_0 + M_r sup_{t\in J} \vee_{s=0}^{t} g(t,s) \leq \|p\|_0 + M_r K = r.$$

Then $\|Fx\|_0 = sup_{t\in J}\|Fx(t)\| \leq r.$ (7)

By (6) and (7), hence $Fx \in Q$ and $FQ \subset Q$ which prove that $F: Q \to Q$. Also $F: Q \to Q$ is weakly sequentially continuous. To see this, let $\{x_n(t)\}$ be sequence in Q weakly convergent to $x(t)$ in E, since Q is closed we have $x(\cdot) \in Q$. Fix $t \in J$, since f satisfies (ii), then we have $f(t, x_n(t))$ converges weakly to $f(t, x(t))$. By the Lebesgue dominated convergence theorem (see assumption (ii)) [22], we have for each $\varphi \in E^*$. $s \in J$

$$\varphi\left(\int_0^t f\big(s, x_n(s)\big)\, d_s g(t,s)\right) = \int_0^t \varphi(f\big(s, x_n(s)\big))\, d_s g(t,s)$$

$$\to \int_0^t \varphi(f\big(s, x(s)\big))\, d_s g(t,s), \forall\, \varphi \in E^*, t \in J$$

Hence

$$|\varphi(Fx_n(t) - Fx(t))| \leq \int_0^t \left|\varphi(f\big(s, x_n(s)\big) - f(s, x(s)))\right| d_s g(t,s) \leq \epsilon$$

and thus F restricted to Q is a weakly sequentially continuous.

Put $V(t) := \{x(t): x(.) \in V\}$ and $(FV)(t) := \{(Fx)(t): x(.) \in V\}$. Suppose that

$V \subset Q$ such that $\bar{V} \subset \overline{conv}(F(V) \cup \{0\})$. We will show that V is weakly relatively compact.

Since Q is bounded and equicontinuous it follows that V is also bounded and equicontinuous. Put

$$v(t) = \beta\big(V(t)\big), for\ \ t \in J.$$

Obviously $V(t) \subset \overline{conv}\big(F\big(V(t)\big) \cup \{0\}\big),\ t \in J$. Using the properties of β we have

$$v(t) \leq \beta(F(V)(t) \cup \{0\}) = \beta\left(F\big(V(t)\big)\right), t \in J.$$

As $V \subset Q$ is equicontinuous, by Lemma 2, the function $t \mapsto v(t)$ is continuous on J. Fix t∈ J, divide the interval $[0,t]$ into n parts $0 = t_0 < t_1 < \ldots < t_n = t,\ t_i - t_{i-1} < \delta,\ i = 1,2,3,\ldots,n$. Put $T_i = [t_{i-1}, t_i]$. In view of Lemma 2 it follows that for each $i \in [1,2,\ldots,n]$ there exists $\tau_i \in T_i$ such that

$$\beta\big(V(T_i)\big) = v(\tau_i),\ i = 1,\ldots,n.$$

By the mean value theorem, we have

$$Fx(t) = p(t) + \int_0^t f\big(s, x(s)\big)\, d\_\{s\}g(t,s) \leq \sum_{i=1}^{n} \int_{T_i} f\big(s, x(s)\big)\, d_s g(t,s)$$

$$\in p(t) + \sum_{i=1}^{n} \mu\big(g(t, T_i)\big)\, \overline{Conv}\{f\big(s, x(s)\big) : x \in V, s \in T_i\}$$

$$\in p(t) + \sum_{i=1}^{n} [g(t,t_i) - g(t,t_{i-1})] \, \overline{Conv} \, f\big(T_i \times V(T_i)\big)$$

$$\subset p(t) + \sum_{i=1}^{n} [g(t,t_i) - g(t,t_{i-1})] \, \overline{Conv} \, f\big(J \times V(T_i)\big) \;\; for\ each\ x \in V$$

Hence

$$FV(t) \subset p(t) + \sum_{i=1}^{n} [g(t,t_i) - g(t,t_{i-1})] \, \overline{Conv} \, f\big(J \times V(T_i)\big) \;\; for\ each\ x \in V$$

by using the properties of the measure of weak noncompactness β we obtain

$$\beta(FV)(t) \subset \sum_{i=1}^{n} [g(t,t_i) - g(t,t_{i-1})] \, \beta(\, f\big(J \times V(T_i)\big))$$

$$\leq \textstyle\sum_{i=1}^{n} [g(t,t_i) - g(t,t_{i-1})] \, h(\, \beta\big(V(T_i)\big)$$

$$\leq \sum_{i=1}^{n} [g(t,t_i) - g(t,t_{i-1})] \, h(\, v(\tau_i))$$

Letting $n \to \infty$ we deduce that

$$\beta(FV)(t) \leq \int_0^t h\big(v(s)\big) \, d_s g(t,s), \qquad t \in J$$

$$v(t) \leq \textstyle\int_0^t h\big(v(s)\big) \, d_s g(t,s), \quad t \in J$$

The condition A_2) implies that the integral inequality above has only trivial solution, i.e. $\beta\big(FV(t)\big) = 0, t \in J$. Thus, $V(t)$ is weakly relatively compact in E. Consequently, Ascoli's theorem proves that V is relatively compact in c_ω. Since all conditions of Theorem 1 are satisfied, then the operator F has at least one fixed point $x(.) \in Q$ and the nonlinear Stieltjes integral equation (1) has at least one weak solution $x(\cdot) \in c_\omega$.

3. Volterra integral equation of fractional order

In this section we show that the Volterra integral equation of fractional order

$$x(t) = x_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s, x(s)\big) \, ds, t \in I \tag{8}$$

can be considered as a special case of the Volterra-Stieltjes integral equation (1).

First, consider, as previously, that the function $g(t,s) = g: \Lambda \to R$. Moreover, we will assume that the function g satisfies the following condition

(v') For $t_1, t_2 \in J, \; t_1 < t_2$, the function $s \to g(t_2,s) - g(t_1,s)$ is nonincreasing on $[0, t_1]$.

Now, we have the following lemmas which proved by Banas *et al*. [7]

Lemma 6: Under assumptions (v') and (vi), for any fixed $s \in J$, the function $t \to g(t,s)$ is nonincreasing on $[s, 1]$.

Lemma 7: Under assumptions (iii), (v') and (vi), the function g satisfies assumption (v). Consider the function g defined by

$$g(t,s) = \frac{t^\alpha - (t-s)^\alpha}{\Gamma(\alpha+1)} \tag{9}$$

Now, we show that the function g satisfies assumptions (iii), (iv), (v') and (vi). Clearly that the function g satisfies assumptions (iii) and (vi). Also, we get

$$d_s g(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} > 0$$

for $0 \leq s < t$. This implies that $s \to g(t,s)$ is increasing on $[0,t]$ for any fixed $t \in J$. Thus the function g satisfies assumption (iv).

To show that g satisfies assumption (v'), let us fix arbitrary $t_1, t_2 \in J,\ t_1 < t_2$. Then we get

$$G(s) = g(t_2, s) - g(t_1, s) = \frac{{t_2}^\alpha - {t_1}^\alpha - (t_2 - s)^\alpha + (t_1 - s)^\alpha}{\Gamma(\alpha+1)},$$

define on $[0, t_1]$. Thus

$$G'(s) = \frac{(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} = \frac{1}{\Gamma(\alpha)}\left[\frac{1}{(t_2 - s)^{1-\alpha}} - \frac{1}{(t_1 - s)^{1-\alpha}}\right]$$

Hence $G'(s) < 0\ for\ s \in [0, t_1)$. This means that g satisfies assumption (v'). And the function g satisfies assumptions (iii)-(vi) in Theorem 2.

Hence, the equation (8) can be written in the form

$$x(t) = x_0 + \int_0^t f(s, x(s))\, d_s g(t,s)$$

Now, we estimate the constants $K,\ N(\epsilon)$ used in our proof. To see this, since the function $s \to g(t,s)$ is nondecreasing on $[0,t]$ for any fixed $t \in J$. Then we have

$$\bigvee_{s=0}^{t} g(t,s) = g(t,t) - g(t,0) = g(t,t) = \frac{t^\alpha}{\Gamma(\alpha+1)},$$

and

$$\bigvee_{s=0}^{t_1} (g(t_2,s) - g(t_1,s)) = \sum_{i=1}^{n} |[g(t_2,s_i) - g(t_1,s_i)] - [g(t_2,s_{i-1}) - g(t_1,s_{i-1})]|$$

$$= \sum_{i=1}^{n} \{[g(t_2,s_{i-1}) - g(t_1,s_{i-1})] - [g(t_2,s_i) - g(t_1,s_i)]\}$$

$$= g(t_1,t_1) - g(t_2,t_1) = \frac{1}{\Gamma(\alpha+1)}[{t_1}^{\alpha} - {t_2}^{\alpha} + (t_2 - t_1)^{\alpha}].$$

Thus

$$K = \sup\left\{\bigvee_{s=0}^{t} g(t,s)\,, t \in J\right\} = \frac{T^{\alpha}}{\Gamma(\alpha+1)}$$

$$N(\epsilon) = \sup\{\bigvee_{s=0}^{t_1}\big(g(t_2,s) - g(t_1,s)\big) : t_1,t_2 \in I, t_2 > t_1, t_2 - t_1 \le \epsilon\}$$

$$= \frac{1}{\Gamma(\alpha+1)}[{t_1}^{\alpha} - {t_2}^{\alpha} + (t_2 - t_1)^{\alpha}].$$

Since

$$\bigvee_{s=t_1}^{t_2}\big(g(t_2,s)\big) = g(t_2,t_2) - g(t_2,t_1) = \frac{1}{\Gamma(\alpha+1)}[{t_2}^{\alpha} - (t_2-t_2)^{\alpha} - {t_2}^{\alpha} + (t_2 - t_1)^{\alpha}]$$

$$= \frac{(t_2 - t_1)^{\alpha}}{\Gamma(\alpha+1)}$$

Then

$$Q = \{x \in C_{\omega} : \|x\|_0 \le r, \|x(t_2) - x(t_1)\| \le \|p(t_2) - p(t_1)\| + \frac{M_r}{\Gamma(\alpha+1)}[|{t_1}^{\alpha} - {t_2}^{\alpha}| + 2(t_2 - t_1)^{\alpha}]\}$$

Since the fractional integral equation (8) is a special case of the equation (1) and by the above properties of the function g, $p(t) = x_0$ we can apply Theorem 2 to formulate the following Corollary concerning the fractional integral equation (8)

Corollary 1: Under the assumption (ii), if

$$\beta(f(J \times X)) \le h(\beta(X))$$

for each $X \subset B_r$, $J \subset I$, such that $J = [0,T], T = min\{a, (\frac{r\Gamma(\alpha+1)}{M_r})^{\frac{1}{\alpha}}\}$, then there exists at least one weak solution $x(.) \in C_{\omega}$ for the fractional integral equation (8) in the nonreflexive Banach space E.

The Volterra integral equation

$$x(t) = x_0 + \int_0^t f\big(s,x(s)\big)\,ds, t \in I \qquad (10)$$

can be considered as a special case of the Volterra-Stieltjes integral equation (1), by consider that the functions $g\,(t,s) = s,\ p(t) = x_0$.

Corollary 2: Under the assumption (ii), if

$$\beta(f(J \times X)) \le h(\beta(X))$$

for each $X \subset B_r$, $J \subset I$, such that $J = [0, d], d = min\{T, \frac{r}{M_r}\}$, then there exists at least one weak solution $x(\cdot) \in C_\omega$ for the fractional integral equation (10) in the nonreflexive Banach space E.

4. Initial Value Problems

As an application for the existence of weak solutions to fractional differential equations in Banach spaces. Since, a Riemann pettis integrable function is sometime called a weakly Riemann integrable function. Also, every weakly continuous function from I into E is Riemann pettis integrable on I (see [23]). It is easy to see that every Riemann Pettis integrable function is Pettis integrable. Thus, the function $x(\cdot) \in C_\omega$ is a solution of the fractional integral equation (8) if and only if $x(\cdot)$ is a solution of the problem (2).

Corollary 3 [14]: Under the assumption (ii), if

$$\beta(f(J \times X)) \leq h(\beta(X))$$

for each $X \subset B_r$, $J \subset I$, such that $J = [0, T], T = min\{a, (\frac{r\Gamma(\alpha+1)}{M_r})^{\frac{1}{\alpha}}\}$, then there exists at least one weak solution $x(.) \in C_\omega$ for the Cauchy problem (2) in the nonreflexive Banach space E.

Finally, for the Cauchy problem (3) in Banach spaces. A function $x(.) \in C_\omega$ is a weak solution of the Volterra integral equation (10) if and only if x is a solution of the Cauchy problem (3). By the equivalent between them.

Corollary 4 [20]: Under the assumption (ii), if

$$\beta(f(J \times X)) \leq h(\beta(X))$$

for each $X \subset B_r$, $J \subset I$, such that $J = [0, d], d = min\{T, \frac{r}{M_r}\}$, then the problem (3) has at least one weak solution $x(.) \in C_\omega$ in the nonreflexive Banach space E.

Corollary 5: [9] If E is a reflexive Banach space, and f is weakly-weakly continuous, then there exists at least one weak solution $x(.) \in C_\omega$ of (3) on I.

5. Conclusion

This research successfully established a more general existence theorem for weakly continuous solutions of the Volterra-Stieltjes integral equation (1) in nonreflexive Banach spaces. This work provides a rigorous theoretical foundation for analyzing the existence of weakly continuous solutions for a broad class of integral and differential equations within the challenging framework of nonreflexive Banach spaces. It employs a powerful fixed-point technique together with the measure of weak noncompactness. Future research could explore the following avenues:

- Numerical and computational methods
- Uniqueness and stability analysis
- Alternative fixed-point theories

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